arXiv:1205.4859v2 [math.RT] 31 Dec 2012

ON A QUESTION OF DRINFELD ON THE WEIL REPRESENTATION I: THE FINITE FIELD CASE

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Abstract. Let F be a finite field of odd cardinality, and let $G = \mathrm{GL}_2(F)$. The group $G \times G \times G$ acts on $F^2 \otimes F^2 \otimes F^2$ via symplectic similitudes, and has a natural Weil representation. Answering a question rasised by V. Drinfeld, we decompose that representation into irreducibles. We also decompose the analogous representation of $\mathrm{GL}_2(A)$, where A is a cubic algebra over F.

Introduction

Let F be a finite field of odd cardinality q, let W be a symplectic vector space over F of dimension $2n$ with form $\langle,\rangle$. The Heisenberg group $H(W)$, attached to W and F, is a set $W \oplus F$ with the group law:

$$(w,t)(w',t') = (w + w', t + t' + \frac{\langle w, w'\rangle}{2}).$$

Let $\mathrm{Sp}(W)$ be the isometry group of $\langle,\rangle$. Define a semi-direct product group $\mathrm{Sp}(W) \ltimes H(W)$ by

$$[g,(w,t)] \cdot [g',(w',t')] = [gg',(w,t) + (g \cdot w',t')].$$

Fix a non-trivial character ψ of F. According to the Stone-Von Neumann theorem, there is only one equivalence class of irreducible complex representation ω_ψ of $H(W)$ with central character ψ. By Weil's celebrated paper [W], in fact ω_ψ is a representation of $\mathrm{Sp}(W) \ltimes H(W)$ in the finite field case. The restriction of ω_ψ to $\mathrm{Sp}(W)$, now is well-known as the *Weil representation*. For our purpose, we extend it to the symplectic similitude group $\mathrm{GSp}(W)$ by setting $\rho = \mathrm{Ind}_{\mathrm{Sp}(W)}^{\mathrm{GSp}(W)} \omega_\psi$, which does not depend on the choice of ψ [cf. [Ge], [Sh]].

The initial question raised by V.Drinfeld, in the finite field case, is understood roughly in the following way. Now assume $\dim_F W = 8$, and fix a canonical basis of W. Let ρ be the Weil representation of $\mathrm{GSp}_8(F)$. Let F^2 be a vector space over F of dimension 2, endowed with the canonical symplectic form. The group $G = \mathrm{GL}_2(F)$ acts on it via symplectic similitudes. Consider $F^2 \otimes F^2 \otimes F^2$ endowed with the symplectic form induced from F^2. It gives rise to a map from $G \times G \times G$ to $\mathrm{GSp}_8(F)$. So we can define a Weil representation π for $G \times G \times G$ via the restriction of ρ. The question is asked about the quotient of π. Does it contain the representations of the form $\sigma \otimes \sigma \otimes \sigma$ for any irreducible representation σ of G?

In this paper, we answer this question and also consider its variant version. To be precise, suppose now that E/F (resp. K/F) is a field extension of degree 2 (resp. 3). Take A to be an étale algebra over F of degree 3, so A is isomorphic to one of the algebras $F \times F \times F, F \times E, K$. We shall construct a homomorphism from $\mathrm{GL}_2(A)$ to $\mathrm{GSp}_8(F)$. If $A = F \times F \times F$, then $\mathrm{GL}_2(A) = \mathrm{GL}_2(F) \times \mathrm{GL}_2(F) \times \mathrm{GL}_2(F)$. This goes back to Drinfeld's question. If $A = F \times E$, then $\mathrm{GL}_2(A) = \mathrm{GL}_2(F) \times \mathrm{GL}_2(E)$. By Weil's Galois descent theory, we construct a quadratic vector space $M = \{\begin{pmatrix} x & \alpha \\ \overline{\alpha} & y \end{pmatrix} | x, y \in F, \alpha \in E\}$ over F of dimension 4, with the quadratic form q defined by the determinant of the matrix. There is a map from $\mathrm{GL}_2(E)$ to $\mathrm{GO}(q)$, which is defined by $h \cdot m = hm\overline{h}^t$, where $h \in H, m \in M$ and $\overline{h}^t$ is the conjugate transpose of h. So $F^2 \otimes M$ is a symplectic vector space over F of dimension 8, and there is a map from $\mathrm{GL}_2(F) \times \mathrm{GL}_2(E)$ to $\mathrm{GSp}_8(F)$. If $A = K$, in this situation, we also need to use Weil's Galois descent to construct a map from $\mathrm{GL}_2(K)$ to $\mathrm{GSp}_8(F)$. For each case, the map $\mathrm{GL}_2(A)$ to $\mathrm{GSp}_8(F)$ leads us to define a

Date: December 2012.
2000 *Mathematics Subject Classification.* 11F27,20C33 (Primary).
Key words and phrases. Weil representation, base change, Shintani lift.

representation $\pi := \rho|_{\mathrm{GL}_2(A)}$. Namely the main purpose of this paper is to obtain the complete decomposition of this representation, and relate it to Shintani lift for GL_2.

Our main results may be formulated as follows. For the group $G = \mathrm{GL}_2(F)$, we write 1_G for the trivial representation of G, St_G for the Steinberg representation of G. Let $T = \{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in G\}$, $B = \{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in G\}$. If χ_1, χ_2 are two characters of $F^\times$, they will bring a character of $\chi_1 \otimes \chi_2$ of T defined by $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \longmapsto \chi_1(a)\chi_2(d)$. We will denote the principal series representation $\mathrm{Ind}_B^G \chi_1 \otimes \chi_2$ of G by π_{χ_1,χ_2}. If (σ, V) is a representation of G and ψ a character of $F^\times$, we define another representation $(\psi \cdot \sigma, V)$ of G by setting $\psi \cdot \sigma(g) = \psi(\det g)\sigma(g)$. Let $\mathrm{Irr}(G)$ denote the class of all irreducible complex representations of the group G. Let L be a field extension of F. By Shintani's work [Sh], one knows that there exists the base-change map $\mathrm{Bc}_{L/F} : \mathrm{Irr}(\mathrm{GL}_2(F)) \longrightarrow \mathrm{Irr}(\mathrm{GL}_2(L))$, which is determined by character equalities.

Theorem (1). *If $A = F \times F \times F$, $\mathrm{GL}_2(A) = \mathrm{GL}_2(F) \times \mathrm{GL}_2(F) \times \mathrm{GL}_2(F)$, then*

$$\pi = \bigoplus_{\sigma \in \mathrm{Irr}(\mathrm{GL}_2(F))} \sigma \otimes \sigma \otimes \sigma \oplus \bigoplus_{\psi \in \mathrm{Irr}(F^\times)} \Big((\psi\cdot\mathrm{St}_G \otimes \psi\cdot 1_G \otimes \psi\cdot 1_G) \oplus (\psi\cdot 1_G \otimes \psi\cdot\mathrm{St}_G \otimes \psi\cdot 1_G) \oplus (\psi\cdot 1_G \otimes \psi\cdot 1_G \otimes \psi\cdot\mathrm{St}_G) \Big).$$

Theorem (2). *If $A = F \times E$, $\mathrm{GL}_2(A) = \mathrm{GL}_2(F) \times \mathrm{GL}_2(E)$, then*

$$\pi = \bigoplus_{\sigma \in \mathrm{Irr}(\mathrm{GL}_2(F))} \sigma \otimes \mathrm{Bc}_{E/F}(\sigma) \oplus \bigoplus_{\psi \in \mathrm{Irr}(F^\times), \Psi \in \mathrm{Irr}(E^\times), \Psi = \psi \circ \mathrm{N}_{E/F}} \Big(\psi \mathrm{St}_G \otimes \Psi \cdot 1_H\Big).$$

Theorem (3). *If $A = K$, $\mathrm{GL}_2(A) = \mathrm{GL}_2(K)$, then*

$$\pi = \bigoplus_{\sigma \in \mathrm{Irr}(\mathrm{GL}_2(F))} \mathrm{Bc}_{K/F}(\sigma).$$

Theorem (1) is obtained mainly by using the method in [A] to decompose a reducible representation. In [A], Andrade considers higher rank groups. Now, let $G = GL_2(F)$. We first formulate the explicit expression about the representation π of $G \times G \times G$ concerned in this case. This can be done by following Gerardin's work on the Weil representation for the group $\mathrm{Sp}_{2n}(F)$ [cf. [Ge]] and also Shinoda's paper for the symplectic similitude group $\mathrm{GSp}_{2n}(F)$[cf. [Sh]]. Then we take two irreducible representations π_1, π_2 of G, and determine the dimension of $\mathrm{Hom}_{1 \times G \times G}\Big(\pi, \pi_1 \otimes \pi_2\Big)$. A key ingredient is that π is in fact a representation of the group $\Big(G \times G \times G\Big) \rtimes S_3$. So if we put $\mathcal{R}_{G\times G\times G}\Big(\pi\Big) = \{\pi_1 \otimes \pi_2 \otimes \pi_3 | \mathrm{Hom}_{G\times G\times G}\Big(\pi, \pi_1 \otimes \pi_2 \otimes \pi_3\Big) \neq 0\}$, then by Clifford's theory, $\mathcal{R}_{G\times G\times G}\Big(\pi\Big)$ is S_3-invariant. This together with the above calculations about the dimensions allows us to prove the Theorem (1).

For the Theorem (2), in this case, put $G = \mathrm{GL}_2(F)$, $H = \mathrm{GL}_2(E)$. Similarly, we first write down the Weil representation π of $G \times H$. We also determine the dimension of the set $\mathrm{Hom}_{1\times H}\Big(\pi, \Pi\Big)$ for any $\Pi \in \mathrm{Irr}(H)$. But it is not enough in this situation. It requires us to formulate the action of G on it and decompose it as a representation of G. We deal with this by checking the irreducible representation Π of H one by one. The main difficulty is when Π is cuspidal. For that case, we use the explicit models given by [A].

The étale algebra $A = K$ is an interesting new case. Set $G = \mathrm{GL}_2(K)$. First we use Weil's Galois descent to construct a map from G to $\mathrm{GSp}(W)$. Let $V = K^2$ be the vector space over K of dimension 2, endowed with a canonical symplectic form $\langle,\rangle_V$. Consider the vector space $\mathbf{W} = K^2 \otimes K^2 \otimes K^2$ over K of dimension 8 with the symplectic form $\langle,\rangle_{\mathbf{W}}$ induced from K^2. Now the Galois group $\mathrm{Gal}(K/F)$ acts on it, at the same time we twist this action by permuting the three variables. Denote by $W = \Big(K^2 \otimes K^2 \otimes K^2\Big)^{\mathrm{Gal}(K/F)}$ the invariant set. One can verify that W is a vector space over F of dimension 8, endowed with symplectic form inherited from $\mathbf{W}$. Go back to the first case. There is a map from $G \times G \times G$ to $\mathrm{GSp}(\mathbf{W})$. Similarly we also consider the Galois action on $G \times G \times G$ twisted by permuting the three variables. The fact is that G is its invariant set. So it allows us to construct a map i from G to $\mathrm{GSp}(W)$. Through this map, we shall define a new Weil representation for the group G. While the explicit realisation of this representation is somehow complex, this causes the difficulty to study

its irreducible components. One point is that the map i sends the Borel subgroup of G to that's of $\mathrm{GSp}(W)$, so by applying the Frobenius Reciprocity Theorem, it is not hard to calculate the dimension $\mathrm{Hom}_G\left(\pi, \pi_{\chi_1,\chi_2}\right)$ for the principal series representation π_{χ_1,χ_2} of G. For the irreducible principal series representation, it is enough and for the trivial representation, we deal with it directly. Then we use a technique-base change, to settle the case of cuspidal representations. We should mention that this method has been used in Gan's paper [Gan] to obtain Howe correspondences for exceptional groups. This still is largely due to the facts that the image of an irreducible cuspidal representation of G will be a principal series representation under the quadratic base change [cf. [Shin]], and the Weil representations are invariant with respect to Shintani lift[cf. [HW]]. So finally it reduces to the previously known case of a non cuspidal representation of $\mathrm{GL}_2(K \times K \otimes E)$.

The paper is arranged as follows. The first section is devoted to giving some notations and recalling some known results that will be used in this paper. Some results of the Weil representations of the groups Sp_{2n} and GSp_{2n} will be reorganized in this section[cf. [Ge], [MVW], [Sh]]. We also recall the classification of the irreducible representations of GL_2[cf. [A], [BH], [P-S]], the lifting of the representations of GL_2[cf. [Shin]] and the behavior of Weil representations with respect to Shintani lift[cf. [HW]]. In the second section, we consider the étale algebra $A = F \times F \times F$. The main part of this section is to determine the dimension $\mathrm{Hom}_{1\times \mathrm{GL}_2(F)\times \mathrm{GL}_2(F)}\left(\pi, \pi_1 \otimes \pi_2\right)$ for any $\pi_1, \pi_2 \in \mathrm{Irr}(\mathrm{GL}_2(F))$, where π is the Weil representation of $\mathrm{GL}_2(F) \times \mathrm{GL}_2(F) \times \mathrm{GL}_2(F)$ in this case. In the third section, we deal with the case $A = F \times E$. In the fourth section, we consider the case $A = K$. We start with the explicit expression of the Weil representation π of $\mathrm{GL}_2(K)$ based on the map from $\mathrm{GL}_2(K)$ to $\mathrm{GSp}_8(F)$. The purpose of this section is to describe the isotypic components of π. We first do it for induced representations, then for cuspidal representations. We put some calculations in two appendices.

Acknowledgements. The paper is one part of the author's Ph.D thesis. The author would like to thank his advisor Professor Guy Henniart for inspiring communications and useful comments.

1. Notation and Preliminaries

1.1. The following notations will be standard through the whole paragraph, and they will be used repeatedly without recalling their meanings:

- F = a finite field with odd cardinality q;
- E = the field extension of F of degree 2;
- K = the field extension of F of degree 3;
- ϕ = a fixed non trivial character of the additive group F;
- ϕ^a = the character of F, defined by $\phi^a(b) := \phi(ab)$ for $b \in F$, $a \in F^\times$;
- X_A = the set of all non trivial irreducible complex representations of an abelian group A;
- $\mathrm{Rep}(G)$ = the category of complex representations of a finite group G;
- $\mathrm{Irr}(G)$ = the class of all irreducible complex representations of a finite group G, up to isomorphism;
- $\check{\sigma}$ = the contragredient representation of σ for any $\sigma \in \mathrm{Rep}(G)$;
- If (σ, V) is a representation of G, then we will denote its G-invariant set by V^G.

1.2. For later use, we recall some notations and properties about the Weil representation of GSp_{2n} (cf. [Ge], [MVW], [Sh]).

Let V be a 2n-dimensional F-vector space, endowed with a non-degenerate symplectic form $\langle\ \rangle$. For each non trivial character ψ of the additive group F, one can associate the Weil representation (ω_ψ, W_ψ) of the metaplectic group $\mathrm{Mp}_{2n}(F)$ (cf. [MVW], chapter 2). The exact sequence

$$1 \longrightarrow \mathbb{C}^\times \longrightarrow \mathrm{Mp}_{2n}(F) \xrightarrow{p} \mathrm{Sp}_{2n}(F) \longrightarrow 1$$

is splitting. Except $n = 1$, $F = \mathbb{F}_3$, the group $\mathrm{Sp}_{2n}(F)$ is perfect, so there exists a unique section-morphism i from $\mathrm{Sp}_{2n}(F)$ to $\mathrm{Mp}_{2n}(F)$, such that $p \circ i = \mathrm{Id}_{\mathrm{Sp}_{2n}(F)}$. In the case $n = 1$, $F = \mathbb{F}_3$, we choose a certain section i in the sense of Gerardin [cf. [Ge], p. 63]. Via the map i, one obtains a representation (ω_ψ, W_ψ) of $\mathrm{Sp}_{2n}(F)$ respect to ψ, called the *Weil representation*. One can extend it as a representation of $\mathrm{GSp}_{2n}(F)$ by setting $\rho_\psi = \mathrm{Ind}_{\mathrm{Sp}_{2n}(F)}^{\mathrm{GSp}_{2n}(F)} \omega_\psi$. It is observed that ρ_ψ is independent on ψ (see [Sh], Theorem 2.15). Hence we could omit ψ, only write ρ briefly.

The study of the Weil representation often involves an explicit model on which the representation is realized. We recall a natural model:

Let $V = V_+ \oplus V_-$ be a complete polarization. Let $\{v_1, \dots, v_n\}$ be a F-basis of V_+, and $\{v_1', \dots, v_n'\}$ its dual basis. Every element $g \in \mathrm{GSp}(V)$ can be written in the following form: $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ where $\alpha \in \mathrm{End}_F(V_+), \beta \in \mathrm{Hom}_F(V_-, V_+), \gamma \in \mathrm{Hom}_F(V_+, V_-), \delta \in \mathrm{End}_F(V_-)$. The group $\mathrm{GSp}(V)$ is generated by the set $\{h(a), u(b), h'(t), \omega\}$ (see [A], p. 163), where $h(a) = \begin{pmatrix} a & 0 \\ 0 & a^\vee \end{pmatrix}$, $a^\vee$ is the contragredient of a; $u(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ for a symmetric morphism $b \in \mathrm{Hom}_F(V_-, V_+)$; $h'(t) = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}, t \in F^\times$; $\omega = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ with $\omega(v_i) = -v_i'$, $\omega(v_i') = v_i$.[1] The Weil representation ρ of $\mathrm{GSp}(V)$ can be realized in the space $W_- = \mathbb{C}[V_- \times X_F]$ of complex functions on $V_- \times X_F$. More precisely the action of $\mathrm{GSp}(V)$ on W_- is determined by the following formulas (cf. [Sh], p. 270):

$$(\rho(h(a))F)(y, \psi) = \chi_q^+(det_{V_+} a) F(a^{\vee -1} y, \psi), \tag{1}$$

$$(\rho(u(b))F)(y, \psi) = \psi(\frac{1}{2}\langle by, y\rangle) F(y, \psi), \tag{2}$$

$$(\rho(\omega)F)(y, \psi) = \gamma(\psi^{-\frac{1}{2}})^{-n} \sum_{z \in V_-} F(z, \psi)\psi(\langle z, \omega^{-1}(y)\rangle), \tag{3}$$

$$(\rho(h'(t))F)(y, \psi) = F(y, \psi^{t^{-1}}), \tag{4}$$

where $y \in V_-, \psi \in X_F, \gamma(\psi) = \sum_{x \in F} \psi(x^2), x_q^+ =$ Legendre symbol $(\frac{}{\mathbb{F}_q})$.

1.3. We sum out some knowledge about the irreducible representations of the group $\mathrm{GL}_2(F)$ and its Borel subgroup B (cf. [BH], chapter 2 and [P-S]).

We write $G = \mathrm{GL}_2(F)$, $B = \{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in G\}$, $N = \{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in G\}$, $T = \{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in G\}$, $M = \{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in G\}$, $Z = \{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in G\}$. Recall that 1_G is the trivial representation of G, and St_G is the Steinberg representation of G. Let θ be a regular character of $E^\times$. The irreducible cuspidal representation of G that corresponds to θ will be denoted by π_θ. If (σ, V) is a representation of G and ψ a character of $F^\times$, we define the representation $(\psi \cdot \sigma, V)$ of G by $\psi \cdot \sigma(g) = \psi(\det g)\sigma(g)$.

Proposition 1.1. *The following is a complete list of the isomorphism classes of the irreducible representations of* G*:*
(1) π_{χ_1, χ_2} *where* $\chi_1 \neq \chi_2$ *are characters of* $F^\times$*;*
(2) $\psi \cdot 1_G$ *where* ψ *ranges over the characters of* $F^\times$*;*
(3) $\psi \cdot \mathrm{St}_G$ *where* ψ *ranges over the characters of* $F^\times$*;*
(4) π_θ *where* θ *ranges over the regular characters of* $E^\times$*.*
The classes in the list are all distinct except that in (1) $\pi_{\chi_1, \chi_2} \simeq \pi_{\chi_2, \chi_1}$*, and in (4)* $\pi_\theta \simeq \pi_{\theta^q}$*.*

Now we investigate the representations of the group B. Let σ_{χ_1, χ_2} be the character of B, which is defined by $\sigma_{\chi_1, \chi_2}\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right) = \chi_1(a)\chi_2(d)$. Let σ be the unique irreducible representation of M of the highest dimension and ψ a character of $F^\times$. Attached to σ and ψ, there is an irreducible representation $\psi \otimes \sigma$ of B, where $(\psi \otimes \sigma)(zm) := \psi(z)\sigma(m)$ for $z \in Z, m \in M$.

[1] In [Sh], the ω is defined as $\omega(v_i) = v_i'$, $\omega(v_i') = -v_i$, but it does not affect the following equation (3). To obtain (3), we use the equality: $\gamma(\psi^{\frac{1}{2}})^{-n} \chi_q^+(-1)^n = \gamma(\psi^{-\frac{1}{2}})^{-n}$.

Proposition 1.2. *The following is a complete list of the isomorphism classes of the irreducible representations of B:*
(1) σ_{χ_1,χ_2} for any pair (χ_1, χ_2) of characters of $F^\times$;
(2) $\psi \otimes \sigma$ for any character ψ of Z.

For convenience use, we describe the decomposition of the restriction to B of any irreducible representation of G.

Proposition 1.3. *(1)* $\mathrm{Res}^G_B(\psi \cdot 1_G) = \sigma_{\psi,\psi}$.
(2) $\mathrm{Res}^G_B(\psi \cdot \mathrm{St}_G) = \sigma_{\psi,\psi} \ \oplus\ \psi^2 \otimes \sigma$.
(3) $\mathrm{Res}^G_B(\pi_{\chi_1,\chi_2}) = \sigma_{\chi_1,\chi_2} \ \oplus\ \sigma_{\chi_2,\chi_1} \oplus \chi_1\chi_2 \otimes \sigma$.
(4) $\mathrm{Res}^G_B(\pi_\theta) = \theta|_{F^\times} \otimes \sigma$.

Proof. See the table in [[A], p. 87]. □

1.4. Let L be the Galois field extension of F of degree n. One knows that there exists the base-change map $\mathrm{Bc}_{L/F} : \mathrm{Irr}(\mathrm{GL}_2(F)) \longrightarrow \mathrm{Irr}(\mathrm{GL}_2(L))$ [cf. [Sh]]. Now we describe the explicit behavior of this map in terms of the classification of the irreducible representations of the group GL_2 in the case $n = 2, 3$.

Proposition 1.4. *(1) If $[L : F] = 2$,*
(i) $\mathrm{Bc}_{L/F}(\pi_{\xi_1,\xi_2}) = \Pi_{\Xi_1,\Xi_2}$ *where $\Xi_i = \xi_i \circ \mathrm{N}_{L/F}$ as characters of $L^\times$, for $i = 1, 2$;*
(ii) $\mathrm{Bc}_{L/F}(\psi \cdot 1_{\mathrm{GL}_2(F)}) = \Psi \cdot 1_{\mathrm{GL}_2(L)}$ *where $\Psi = \psi \circ \mathrm{N}_{L/F}$ as characters of $L^\times$;*
(iii) $\mathrm{Bc}_{L/F}(\psi \cdot \mathrm{St}_{\mathrm{GL}_2(F)}) = \Psi \cdot \mathrm{St}_{\mathrm{GL}_2(L)}$ *where $\Psi = \psi \circ \mathrm{N}_{L/F}$ as characters of $L^\times$;*
(iv) $\mathrm{Bc}_{L/F}(\pi_\theta) = \Pi_{\theta,\theta^q}$.
(2)If $[L : F] = 3$,
(i) $\mathrm{Bc}_{L/F}(\pi_{\xi_1,\xi_2}) = \Pi_{\Xi_1,\Xi_2}$ *where $\Xi_i = \xi_i \circ \mathrm{N}_{L/F}$ as characters of $L^\times$,for $i = 1, 2$;*
(ii) $\mathrm{Bc}_{L/F}(\psi \cdot 1_{\mathrm{GL}_2(F)}) = \Psi \cdot 1_{\mathrm{GL}_2(L)}$ *where $\Psi = \psi \circ \mathrm{N}_{L/F}$ as characters of $L^\times$;*
(iii) $\mathrm{Bc}_{L/F}(\psi \cdot \mathrm{St}_{\mathrm{GL}_2(F)}) = \Psi \cdot \mathrm{St}_{\mathrm{GL}_2(L)}$ *where $\Psi = \psi \circ \mathrm{N}_{L/F}$ as characters of $L^\times$;*
(iv) $\mathrm{Bc}_{L/F}(\pi_\theta) = \Pi_\Theta$ *where* $[F_1 : F] = 2, [L_1 : L] = 2, L_1 \supseteq F_1, \theta \in \mathrm{Irr}(F_1^\times) - \mathrm{Irr}(F^\times), \Theta \in \mathrm{Irr}(L_1^\times) - \mathrm{Irr}(L^\times)$, *and* $\Theta = \theta \circ \mathrm{N}_{L_1/F_1}$ *as characters of* $L_1^\times$.

Proof. See [[Shin], Section 4, p. 410—414]. □

1.5. In the article [Gy], Gyoja generalizes the map of base change of GL_2 to connected linear algebraic group, called *Shintani lift*. We will recall his results below. In addition, we also present one main result in [HW] about the behavior of the Weil representations with respect to Shintani lift.

Let $\overline{F}$ be a fixed algebraic closure of F with Frobenius map σ. Let $\mathbf{G}$ be a connected linear algebraic group over F. Denote by F_i the σ^i-fixed points of $\overline{F}$. Let Y be a set on which there exists a σ-action, we denote the set of σ-fixed points by Y_σ. Denote by $\mathbf{G}(F_i)$ the F_i-geometric points of $\mathbf{G}$ and $C(\mathbf{G}(F_i))$ the set of complex valued class functions of $\mathbf{G}(F_i)$. Fix a positive integer m. Via the map $\mathrm{Gal}(\overline{F}/F) \twoheadrightarrow \mathrm{Gal}(F_m/F)$, we view the Frobenius element σ as one generator for the group $\mathrm{Gal}(F_m/F)$. For $0 \le i \le m-1$, let us denote by $\sigma^i \ltimes \mathbf{G}(F_m)$, the subset of the semi-direct product $\mathrm{Gal}(F_m/F) \ltimes \mathbf{G}(F_m)$ consisting of elements (σ^i, g) for any $g \in \mathbf{G}(F_m)$. In the article [Gy], Gyoja constructs the norm map N_i in the following way:

$$N_i : \sigma^i \ltimes \mathbf{G}(F_m) \longrightarrow \mathbf{G}(\overline{F});$$
$$[\sigma^i, x] \longmapsto \alpha(x)\Big(x\sigma^i(x)\cdots\sigma^{i(\frac{m}{d}-1)}(x)\Big)\alpha(x)^{-1},$$

where $\alpha(x)$ is an element in $\mathbf{G}(\overline{F})$ such that $\alpha(x)^{-1}\sigma^d\big(\alpha(x)\big) = x\sigma^i(x)\cdots\sigma^{i(t-1)}(x)$ and d, t are integers given by $d = (m, i)$ and $ti \equiv d(\mod m)$, here (m, i) denote the greatest common divisor of natural numbers m and i.

The above norm map induces a bijection from the set of $\mathbf{G}(F_m)$-conjugacy classes of $\sigma^i \ltimes \mathbf{G}(F_m)$ onto the set of conjugacy classes of $\mathbf{G}(F_m)_{\sigma^i} = \mathbf{G}(F_{(m,i)})$. Through the norm map, one defines the i-restriction map from $C(\mathrm{Gal}(F_m/F) \ltimes \mathbf{G}(F_m))$ to $C(\mathbf{G}(F_{(m,i)}))_\sigma$ such that $(i\text{–}res(f)) \circ \mathrm{N}_i = f|_{\sigma^i \ltimes \mathbf{G}(F_m)}$ for any $f \in C(\mathrm{Gal}(F_m/F) \ltimes \mathbf{G}(F_m))$.

Lemma 1.5. *(i) For any $f, g \in C(\mathbf{G}(F_m)_{\sigma^i})$, we have $\langle f, g\rangle_{\mathbf{G}(F_m)_{\sigma^i}} = \langle f \circ N_i, g \circ N_i\rangle_{\sigma^i \ltimes \mathbf{G}(F_m)}$, where $\langle f, g\rangle_{\mathbf{G}(F_{(m,i)})} := \frac{1}{|\mathbf{G}(F_{(m,i)})|}\sum_{x \in \mathbf{G}(F_{(m,i)})} f(x)\overline{g(x)}$ and $\langle f \circ N_i, g \circ N_i\rangle_{\sigma^i \ltimes \mathbf{G}(F_m)} := \frac{1}{|\sigma^i \ltimes \mathbf{G}(F_m)|}\sum_{t \in \mathbf{G}(F_m)} f(N_i(\sigma^i, t))\overline{g(N_i(\sigma^i, t))}$.*
(ii) The i-restriction defines an isomorphism $C(\mathrm{Gal}(F_m/F)\mathbf{G}(F_m)) \simeq \oplus_{i=0}^{m-1} C(\mathbf{G}(F_m)_{\sigma^i})_\sigma$.

Proof. See [[Gy], p.1 Introduction and p.11 Corollary 3.3]. □

Lemma 1.6. *Let $\mathbf{H}$ be a connected closed subgroup of $\mathbf{G}$ defined over F. Then the following diagram is commutative*

$$\begin{array}{ccc} C(\operatorname{Gal}(F_m/F) \ltimes \mathbf{G}(F_m)) & \xrightarrow{\text{Res}} & C(\operatorname{Gal}(F_m/F) \ltimes \mathbf{H}(F_m)) \\ \downarrow{\scriptstyle i-res} & & \downarrow{\scriptstyle i-res} \\ C(\mathbf{G}(F_{(m,i)}))_\sigma & \xrightarrow{\text{Res}} & C(\mathbf{H}(F_{(m,i)}))_\sigma \end{array}$$

Proof. See [[Gy], p. 12, Lemma 3.6]. □

Now let $\mathbf{G} = \mathbf{GSp}_V$ the algebraic group of isometry of similitudes of a symplectic vector space V over F. For $0 \leq i \leq m-1$, write $\Xi_{F_{(m,i)}}$ for the Weil representation of $\mathbf{GSp}_V(F_{(m,i)})$

Theorem 1.7. *There exists a unique representation $\widetilde{\Xi_{F_m}}$ of the group $\operatorname{Gal}(F_m/F)\mathbf{GSp}_V(F_m)$ such that i-res($\widetilde{\Xi_{F_m}}$)= $\Xi_{F_{(m,i)}}$ for $0 \leq i \leq m-1$.*

Proof. See the main theorem in [HW]. □

2. The decomposition of the Weil representation of $\mathrm{GL}_2(F)\times\mathrm{GL}_2(F)\times\mathrm{GL}_2(F)$

2.1. We give some notations, and formulate the main representation that will be studied in this section.

In this section, we use the following notations: $G = \mathrm{GL}_2(F)$, $H = G \times G$, $B = \{\begin{pmatrix} a & b \\ 0 & d\end{pmatrix} \in G\}$, $N = \{\begin{pmatrix} 1 & b \\ 0 & 1\end{pmatrix} \in G\}$, $T = \{\begin{pmatrix} a & 0 \\ 0 & d\end{pmatrix} \in G\}$, $Z = \{\begin{pmatrix} a & 0 \\ 0 & a\end{pmatrix} \in G\}$; $h(r) = \begin{pmatrix} r & 0 \\ 0 & r^{-1}\end{pmatrix}$, $u(b) = \begin{pmatrix} 1 & b \\ 0 & 1\end{pmatrix}$, $h'(t) = \begin{pmatrix} 1 & 0 \\ 0 & t\end{pmatrix}$, $\omega' = \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}$, $\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}$ elements in G, S_3 = the permutation group of 3 variables.

Let V be a vector space over F of dimension 2, endowed with a non-degenerate symplectic form $\langle, \rangle$. Let $\{e_1, e_2\}$ be a canonical basis of V i.e. $\langle e_1, e_2\rangle = 1$, $\langle e_2, e_1\rangle = -1$. We attach the vector space $V^{\otimes 3} = V \otimes_F V \otimes_F V$ with the natural symplectic form $\langle,\rangle \otimes \langle,\rangle \otimes \langle,\rangle$ induced from V, so there exists a homomorphism p from $\big(\mathrm{GSp}(V) \times \mathrm{GSp}(V) \times \mathrm{GSp}(V)\big) \rtimes S_3$ to $\mathrm{GSp}(V^{\otimes 3})$. The above group S_3 acts on $V \otimes V \otimes V$ by permutation. By the fixed basis $\{e_1, e_2\}$ of V and $\{e_i \otimes e_j \otimes e_k | 1 \leq i, j, k \leq 2\}$ of $V^{\otimes 3}$, we could identify the group G with $\mathrm{GL}(V)$, and also the group $\mathrm{GSp}_8(F)$ with $\mathrm{GSp}(V^{\otimes 3})$.

Let ρ be the Weil representation of $\mathrm{GSp}(V^{\otimes 3})$. Through the above morphism p, it gives rise to a representation π' of the group $\big(\mathrm{GSp}(V)\times\mathrm{GSp}(V)\times\mathrm{GSp}(V)\big)\rtimes S_3$. Let π denote the restriction representation of π' to the subgroup $\mathrm{GSp}(V) \times \mathrm{GSp}(V) \times \mathrm{GSp}(V)$. Write ${}_+V^{\otimes 3} = \{x \in V^{\otimes 3} | x \in Fe_1 \otimes V \otimes V\}$, ${}_-V^{\otimes 3} = \{y \in V^{\otimes 3} | y \in Fe_2 \otimes V \otimes V\}$. Every element $y \in {}_-V^{\otimes 3}$ has the form $y = \sum_{j,k=1}^{2} a_{j,k} e_2 \otimes e_j \otimes e_k$, which corresponds to a matrix $m = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}$. So we could identify ${}_-V^{\otimes 3}$ with the matrix ring $M_2(F)$ as vector space over F. The representation π of the group $G \times G \times G$ can be realized in the vector space $W = \mathbb{C}[M_2(F) \times X_F]$ of complex functions on $M_2(F) \times X_F$.

Proposition 2.1. *The representation $(\pi, G \times G \times G, W)$ is given by the following formulas:*

$$(\pi[h(a), 1, 1] f)(m, \psi) = f(am, \psi), \tag{5}$$

$$(\pi[u(b), 1, 1] f)(m, \psi) = \psi(b \det(m)) f(m, \psi), \tag{6}$$

$$(\pi[h'(t), 1, 1] f)(m, \psi) = f(m, \psi^{t^{-1}}), \tag{7}$$

$$(\pi[\omega, 1, 1] f)(m, \psi) = q^{-2} \sum_{n \in M_2(F)} \psi(B(m, n))\, f(n, \psi) \tag{8}$$

$$(\pi[1, g_2, g_3] f)(m, \psi) = f(\det(g_2g_3)g_2^{-1} m g_3^{-1t}, \psi^{det(g_2g_3)^{-1}}) \tag{9}$$

where $g_2, g_3 \in G, m \in M_2(F)$, g_3^t= *the transpose of* g_3, $B(m,n) = m_{11}n_{22} + m_{22}n_{11} - m_{12}n_{21} - m_{21}n_{12}$ *for* $m = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22}\end{pmatrix}$, $n = \begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22}\end{pmatrix} \in M_2(F)$.

Proof. The formulas (5)—(8) come directly from the formulas (1)—(4) in Section 1. Consider now the formula (9). Recall, for $g \in G$, $g \cdot e_1 := (e_1, e_2) g \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $g \cdot e_2 := (e_1, e_2) g \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. By the fixed basis $\{e_2 \otimes e_j \otimes e_k | 1 \leq j, k \leq k \leq 2\}$, we obtain $g_2 \otimes g_3 \cdot m := g_2 m g_3^t$ for $g_2, g_3 \in G, m \in M_2(F)$. Then

$$\big(\pi[1, g_2, g_3] f\big)(m, \psi) = \rho\Big(\begin{pmatrix} 1 & 0 \\ 0 & \det(g_2 g_3)\end{pmatrix}\begin{pmatrix} g_2 \otimes g_3 & 0 \\ 0 & \det(g_2g_3)^{-1} g_2 \otimes g_3\end{pmatrix}\Big) f(m, \psi)$$

$$\overset{\text{equality (4) in section 1}}{=} \rho\Big(\begin{pmatrix} g_2 \otimes g_3 & 0 \\ 0 & \det(g_2g_3)^{-1} g_2 \otimes g_3\end{pmatrix}\Big) f(m, \psi^{\det(g_2g_3)^{-1}})$$

$$\overset{\text{equality (1) in section 1}}{=} \chi_q^+\big(\det(g_2 \otimes g_3)^2\big) f(g_2^{-1} \otimes g_3^{-1} \det(g_2g_3) \cdot m, \psi^{\det(g_2g_3)^{-1}}) = f(\det(g_2g_3) g_2^{-1} m g_3^{-1t}, \psi^{\det(g_2g_3)^{-1}}).$$

□

2.2. To decompose the representation π, it involves to describe the $1 \times G \times G$-invariant part of the vector space W.

We consider the set $\mathcal{S} = \{(\pi_1, \pi_2) |$ for $i = 1, 2, (\pi_i, V_i) \in \mathrm{Rep}(G)$ such that $(\pi_1 \otimes \pi_2)|_Z = \mathrm{Id}_{V_1 \times V_2}\}$. For each pair $(\pi_1, \pi_2) \in \mathcal{S}$, it determines a representation $(\pi_1 \otimes \pi_2, V_1 \otimes V_2)$ of the group H. We write $\mathrm{Irr}_0(H)$ for the set of isomorphism classes of all irreducible representations $(\pi_1 \otimes \pi_2, V_1 \otimes V_2)$ of H for which $(\pi_1, \pi_2) \in \mathcal{S}$.

Now we concentrate on the decomposition of the representation $(\pi, G \times G \times G, W)$. Following the method in [A], we first associate a representation $(\pi_0', G, W[\pi_1 \otimes \pi_2])$ for any representation $\pi_1 \otimes \pi_2 \in \mathrm{Irr}_0(H)$, where the vector space $W[\pi_1 \otimes \pi_2]$ consists of all functions $f : M_2(F) \times X_F \longrightarrow V_1 \otimes V_2$ such that :

$$f(\det(g_1g_2) g_1^{-1} m g_2^{-1t}, \psi^{det(g_1g_2)^{-1}}) = (\pi_1(g_1) \otimes \pi_2(g_2)) f(m, \psi), \tag{10}$$

for $(g_1, g_2) \in H, m \in M_2(F), \psi \in X_F$; and the action of G on $W[\pi_1 \otimes \pi_2]$ is given by the formulas (5)— (8) in Proposition 2.1 .

Proposition 2.2. *For the representation* $(\pi, G, G \times G \times G)$*, we have the following decomposition:*

$$W = \bigoplus_{\pi_1 \otimes \pi_2 \in Irr_0(H)} W[\pi_1 \otimes \pi_2] \otimes V_1 \otimes V_2.$$

Proof. The Weil representation (π, W) of $G \times G \times G$ has the following decomposition $W = \oplus_{\pi_1 \otimes \pi_2 \in Irr_0(H)} W_{\pi_1 \otimes \pi_2} \otimes V_1 \otimes V_2$ and we have the identification of vector spaces $W_{\pi_1 \otimes \pi_2} \simeq [W \otimes (V_1 \otimes V_2)^\vee]^{1 \times G \times G} \simeq \mathrm{Hom}_{1 \times G \times G}(W, \mathbb{C} \otimes V_1 \otimes V_2) \simeq W[V_1 \otimes V_2]$. □

Recall the Cartan involution $\theta : G \longrightarrow G; g \longmapsto (g^t)^{-1}$. It is well-known that $\check{\sigma} \simeq \sigma \circ \theta$ for $\sigma \in \mathrm{Irr}(G)$. By using this special result for G, and also the isomorphism of vector spaces $\lambda : V_1 \otimes V_2^\star \longrightarrow \mathrm{Hom}_{\mathbb{C}}(V_2, V_1); v_1 \otimes v_2^\star \longmapsto \varphi_{v_1 \otimes v_2^\star}$ where $\varphi_{v_1 \otimes v_2^\star}(v_2) = \langle v_2^\star, v_2 \rangle v_1$, we could obtain the isomorphism of representations of H: $(\pi_1 \otimes \pi_2, V_1 \otimes V_2) \simeq ([\pi_1, \pi_2 \circ t], \mathrm{Hom}_{\mathbb{C}}(V_2, V_1))$ where $[\pi_1, \pi_2 \circ t](g_1, g_2)(\varphi) = \pi_1(g_1) \circ \varphi \circ \pi_2(g_2^t)$ for $g_1, g_2 \in G, \varphi \in \mathrm{Hom}_{\mathbb{C}}(V_2, V_1)$. Now let $W[\pi_1, \pi_2]$ be a vector space consisting of all functions $f : M_2(F) \times X_F \longrightarrow \mathrm{Hom}_{\mathbb{C}}(V_2, V_1)$ such that:

$$f(\det(g_1g_2) g_1^{-1} m g_2^{-1t}, \psi^{det(g_1g_2)^{-1}}) = \pi_1(g_1) \circ f(m, \psi) \circ \pi_2(g_2^t). \tag{11}$$

Hence the following proposition is straightforward:

Proposition 2.3. *Let* $\pi_1 \otimes \pi_2 \in \mathrm{Irr}_0(H)$*. Then* $(\pi_0', G, W[\pi_1 \otimes \pi_2]) \simeq (\pi_0, G, W[\pi_1, \pi_2])$*, where the action of* $\pi_0(g)$ *on* $W[\pi_1, \pi_2]$ *is given by the formulas (5)—(8) in Proposition 2.1, for any* $g \in G$*.*

2.3. We continue the above discussion, and determine the dimension of the vector space $W[\pi_1,\pi_2]$.

For $\pi_1\otimes\pi_2\in \mathrm{Irr}_0(H)$, we write $W[\pi_1,\pi_2](\xi)=\{f(\xi)|f\in W[\pi_1,\pi_2],\xi\in M\times X_F\}$. Now we define an H-action on the set $M_2(F)\times X_F$ as follows:

$$(g_1,g_2)(m,\psi):=(\det(g_1^{-1}g_2^{-1})g_1mg_2^t,\psi^{\det(g_1g_2)}), \tag{12}$$

where $(g_1,g_2)\in H,\psi\in X_F, m\in M_2(F)$.

It is observed that $W[\pi_1,\pi_2](\xi)=\mathrm{Fix}_{\mathrm{Hom}_{\mathbb{C}}(V_2,V_1)}(\mathrm{Stab}_H(\xi))$ for $\xi\in M_2(F)\times X_F$, more precisely

$$W[\pi_1,\pi_2](\xi)=\{\varphi:V_2\longrightarrow V_1|\pi_1(g_1)\circ\varphi=\varphi\circ\pi_2(g_2^{t^{-1}})\quad \forall g_1,g_2\in \mathrm{Stab}_H(\xi)\}. \tag{13}$$

Let's determine the H-orbits in $M_2(F)\times X_F$. They are of the following three kinds:

(i) Orbit $\{\xi_a\}$ where $\xi_a=(\begin{pmatrix}1&0\\0&1\end{pmatrix},\phi^a)$ for any $a\in F^\times$;

(ii) Orbit $\{\eta\}$ where $\eta=(\begin{pmatrix}1&0\\0&0\end{pmatrix},\phi)$;

(iii) Orbit $\{\delta\}$ where $\delta=(\begin{pmatrix}0&0\\0&0\end{pmatrix},\phi)$.

And by straightforward calculation, the corresponding stabilizer of the given representative element in each orbit has the following form:

(i) $\mathrm{Stab}_H(\xi_a)=\{(g,g^{-1t})|g\in G\}$;
(ii) $\mathrm{Stab}_H(\eta)=\{(sn_1,s^{-1}n_2)|s\in T,n_1,n_2\in N\}$;
(iii) $\mathrm{Stab}_H(\delta)=\{(g_1,g_2)|$ where $g_1,g_2\in G$ and $\det(g_1g_2)=1\}$.

To obtain the dimension of the vector space $W[\pi_1,\pi_2]$, we state the lemma:

Lemma 2.4. *(1) $W[\pi_1,\pi_2](\xi_a)\neq 0$ iff $\pi_1\simeq\pi_2$, in this case $\dim_{\mathbb{C}} W[\pi_1,\pi_2](\xi_a)=1$;*
(2) $W[\pi_1,\pi_2](\eta)=0$ except the following cases:

(a) $\dim_{\mathbb{C}} W[\pi_{\chi_1,\chi_2},\pi_{\chi_1,\chi_2}](\eta)=2$,
(b) $\dim_{\mathbb{C}} W[\psi\cdot 1_G,\psi\cdot 1_G](\eta)=1$,
(c) $\dim_{\mathbb{C}} W[\psi\cdot \mathrm{St}_G,\psi\cdot \mathrm{St}_G](\eta)=1$,
(d) $\dim_{\mathbb{C}} W[\psi\cdot 1_G,\psi\cdot \mathrm{St}_G](\eta)=1$,
(e) $\dim_{\mathbb{C}} W[\psi\cdot \mathrm{St}_G,\psi\cdot 1_G](\eta)=1$,

for the characters $\chi_1\neq\chi_2$, ψ of $F^\times$;
(3) $W[\pi_1,\pi_2](\delta)=0$ except $\pi_1=\pi_2=\psi\cdot 1_G$, in that case $\dim_{\mathbb{C}} W[\psi\cdot 1_G,\psi\cdot 1_G](\delta)=1$, for any character ψ of $F^\times$.

Proof. (1) $W[\pi_1,\pi_2](\xi_a)=\{\varphi:V_2\longrightarrow V_1|\pi_1(g_1)\circ\varphi=\varphi\circ\pi_2(g_2^{-1t})$ for $(g_1,g_2)\in \mathrm{Stab}_H(\xi_a)\}\simeq \mathrm{Hom}_G(V_2,V_1)$.
(2) Note that $W[\pi_1,\pi_2](\eta)\simeq \mathrm{Hom}_T(V_2^N,V_1^N)$. Therefore, $W[\pi_1,\pi_2](\eta)=0$ unless π_1,π_2 both are induced representations. In the later case, for $(\pi_{\chi_1,\chi_2},V)=\mathrm{Ind}_B^G\,\chi_1\otimes\chi_2$ with $\chi_1,\chi_2\in\mathrm{Irr}(F^\times)$. The vector space V^N is generated by functions f_{χ_1,χ_2} and g_{χ_1,χ_2}, where $\mathrm{supp}\, f_{\chi_1,\chi_2}=B$, $\mathrm{supp}\, g_{\chi_1,\chi_2}=B\omega'N$ and they satisfy $f_{\chi_1,\chi_2}(tn)=\chi_1\otimes\chi_2(t)$, $g_{\chi_1,\chi_2}(tn_1\omega'n_2)=\chi_1\otimes\chi_2(t)$ for $t\in T,n,n_1,n_2\in N$. The action of T on $\{f_{\chi_1,\chi_2},g_{\chi_1,\chi_2}\}$ is simply by the formulas:

$$t\cdot f_{\chi_1,\chi_2}=\chi_1\otimes\chi_2(t)f_{\chi_1,\chi_2},\quad\text{ and }\quad t\cdot g_{\chi_1,\chi_2}=\chi_2\otimes\chi_1(t)g_{\chi_1,\chi_2} \text{ for } t\in T.$$

Thus, $\dim_{\mathbb{C}}\mathrm{Hom}_T(\pi^N_{\chi_1,\chi_2},\pi^N_{\chi_1,\chi_2})=2$ for $\chi_1\neq\chi_2\in\mathrm{Irr}(F^\times)$, so (a) follows. On the other hand $\dim_{\mathbb{C}}\mathrm{Hom}_T(\pi^N_{\psi,\psi},\pi^N_{\psi,\psi})=4$ for $\psi\in\mathrm{Irr}(F^\times)$. Obviously $\mathrm{St}_G^N=\mathbb{C}(qf_{1_G,1_G}-g_{1_G,1_G})$ and $1_G^N=\mathbb{C}(f_{1_G,1_G}+g_{1_G,1_G})$, so we could obtain (b)—(e).
(3) $W[\pi_1,\pi_2](\delta)=\{\varphi:V_2\longrightarrow V_1|\pi_1(g_1)\circ\varphi=\varphi\circ\pi_2(g_2^{-1t})\quad (g_1,g_2)\in\mathrm{Stab}_H(\delta)\}=\{\varphi:V_2\longrightarrow V_1|\pi_1(g_1)\circ\varphi=\varphi\circ\pi_2(g_2)$ where $g_1,g_2\in G$, $\det g_1=\det g_2\}$. By considering the set $\{g\in G|\det(g)=1\}$, we know $W[\pi_1,\pi_2](\delta)=0$, unless, $\pi_1=\psi_1\circ\det$ and $\pi_2=\psi_2\circ\det$. In the later case, $W[\pi_1,\pi_2](\delta)\simeq\mathrm{Hom}_{F^\times}(\psi_2,\psi_1)$, so the result follows. □

Corollary 2.5. *For any irreducible representation $\pi_1\otimes\pi_2\in\mathrm{Irr}_0(H)$, the dimension of the representation $(\pi_0,G,W[\pi_1,\pi_2])$ is presented as follows:*

(i) $\dim_{\mathbb{C}} W[\pi_{\chi_1,\chi_2},\pi_{\chi_1,\chi_2}]=q+1$,
(ii) $\dim_{\mathbb{C}} W[\psi\cdot\mathrm{St}_G,\psi\cdot\mathrm{St}_G]=q$,
(iii) $\dim_{\mathbb{C}} W[\psi\cdot 1_G,\psi\cdot 1_G]=q+1$,
(iv) $\dim_{\mathbb{C}} W[\pi_\theta,\pi_\theta]=q-1$,

(v) $\dim_{\mathbb{C}} W[\psi\cdot \mathrm{St}_G, \psi\cdot 1_G)=1$,
(vi) $\dim_{\mathbb{C}} W[\psi\cdot 1_G, \psi\cdot \mathrm{St}_G)=1$,

for the characters $\chi_1 \neq \chi_2, \psi$ of $F^\times$, the regular character θ of $E^\times$. And the above list are all representations $\pi_1\otimes\pi_2 \in \mathrm{Irr}_0(H)$, such that $W[\pi_1,\pi_2]\neq 0$.

2.4. We have already calculated the dimension of the vector space $W[\pi_1,\pi_2]$, and it is enough to obtain the main theorem in this section.

Theorem 2.6. *The decomposition of the representation $(\pi, G\times G\times G, W)$ has the following form:*

$$\pi=\bigoplus_{\sigma\in\mathrm{Irr}(G)}\Big(\sigma\otimes\sigma\otimes\sigma\Big)\oplus\bigoplus_{\psi\in\mathrm{Irr}(F^\times)}\Big((\psi\cdot\mathrm{St}_G\otimes\psi\cdot 1_G\otimes\psi\cdot 1_G)\oplus(\psi\cdot 1_G\otimes\psi\cdot\mathrm{St}_G\otimes\psi\cdot 1_G)\oplus(\psi\cdot 1_G\otimes\psi\cdot 1_G\otimes\psi\cdot\mathrm{St}_G)\Big).$$

Proof. Since $\pi=\mathrm{Res}^{(G\times G\times G)\rtimes S_3}_{G\times G\times G}\pi'$, by Clifford's theory, the representation π could be written as the direct sum of the following three kinds of forms: (1) $\tau_0\otimes\tau_0\otimes\tau_0$ for $\tau_0\in\mathrm{Irr}(G)$; (2) $\tau_1\otimes\tau_1\otimes\tau_2+\tau_1\otimes\tau_2\otimes\tau_1+\tau_2\otimes\tau_1\otimes\tau_1$ for $\tau_1\neq\tau_2\in\mathrm{Irr}(G)$; (3) $\tau_0'\otimes\tau_1'\otimes\tau_2'+\tau_0'\otimes\tau_2'\otimes\tau_1'+\tau_1'\otimes\tau_0'\otimes\tau_2'+\tau_1'\otimes\tau_2'\otimes\tau_0'+\tau_2'\otimes\tau_0'\otimes\tau_1'+\tau_2'\otimes\tau_1'\otimes\tau_0'$ for three different representations τ_0',τ_1',τ_2' in $\mathrm{Irr}(G)$. By comparing this with the results in Corollary 2.5, we obtain the theorem. □

3. The decomposition of the Weil representation of $\mathrm{GL}_2(F)\times\mathrm{GL}_2(E)$

3.1. We first give some notations, and formulate the representation concerned in this section.

In this section, we denote by: $G=\mathrm{GL}_2(F)$, $B=\{\begin{pmatrix}a&b\\0&d\end{pmatrix}\in G\}$, $N=\{\begin{pmatrix}1&b\\0&1\end{pmatrix}\in G\}$, $T=\{\begin{pmatrix}a&0\\0&d\end{pmatrix}\in G\}$, $Z=\{\begin{pmatrix}a&0\\0&a\end{pmatrix}\in G\}$; $H=\mathrm{GL}_2(E)$, $B'=\{\begin{pmatrix}a&b\\0&d\end{pmatrix}\in H\}$, $N'=\{\begin{pmatrix}1&b\\0&1\end{pmatrix}\in H\}$, $T'=\{\begin{pmatrix}a&0\\0&d\end{pmatrix}\in H\}$, $Z'=\{\begin{pmatrix}a&0\\0&a\end{pmatrix}\in H\}$; $h(r)=\begin{pmatrix}r&0\\0&r^{-1}\end{pmatrix}$, $u(b)=\begin{pmatrix}1&b\\0&1\end{pmatrix}$, $h'(t)=\begin{pmatrix}1&0\\0&t\end{pmatrix}$, $\omega'=\begin{pmatrix}0&1\\1&0\end{pmatrix}$, $\omega=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$ elements in G or H; $\mathrm{Gal}(E/F)=\langle\sigma\rangle$. If $h\in H$ or $M_2(E)$, we will denote its conjugate by h^σ or $\overline{h}$, its transpose by h^t, and $h^\star:=\overline{h}^t$.

Let V be the vector space over F of dimension 2, endowed with a symplectic form $\langle,\rangle$. Let $\{e_1,e_2\}$ be a symplectic basis of V. Consider the E-vector space $V_E=E\otimes_F V$, endowed with the symplectic form $\langle,\rangle_{V_E}$, induced from V. Define a $\mathrm{Gal}(E/F)$-action on V_E by

$$\mathrm{Gal}(E/F)\times E\otimes_F V\longrightarrow E\otimes_F V;(\sigma,\sum_i t_i\otimes e_i)\longmapsto\sum_i t_i^\sigma\otimes e_i.$$

Now let $\mathbb{W}=V_E\otimes_E V_E$, we associate $\mathbb{W}$ a symmetric form $\langle,\rangle_{\mathbb{W}}=\langle,\rangle_{V_E}\otimes\langle,\rangle_{V_E}$. On $\mathbb{W}$, we will consider the twisted Galois action defined by

$$\mathrm{Gal}(E/F)\times\mathbb{W}\longrightarrow\mathbb{W};(\sigma,w=\sum_i u_i\otimes v_i)\longmapsto{}^\sigma w=\sum_i v_i^\sigma\otimes u_i^\sigma.$$

We will let $\mathbb{W}_0$ denote the set $\{w\in\mathbb{W}|{}^\sigma w=w\}$. One can check that the symmetric form over the F-vector space $\mathbb{W}_0$ is a F-symmetric form, denoted by $(,)_{\mathbb{W}_0}$. And we denote its associative quadratic form by q, defined by

$$(w_0,w_0')_{\mathbb{W}_0}=q(w_0+w_0')-q(w_0)-q(w_0')\text{ for }w_0,w_0'\in\mathbb{W}_0.$$

By calculation, each $w_0\in\mathbb{W}_0$ may be expressed in the form

$$w_0=xe_1\otimes e_1+\alpha e_1\otimes e_2+\overline{\alpha}e_2\otimes e_1+ye_2\otimes e_2\text{ for }x,y\in F;\alpha\in E.$$

Every element w_0 corresponds to a matrix $\begin{pmatrix}x&\alpha\\\overline{\alpha}&y\end{pmatrix}$. So we identify $\mathbb{W}_0$ with $M=\{\begin{pmatrix}x&\alpha\\\overline{\alpha}&y\end{pmatrix}|x,y\in F,\alpha\in E\}$. The symmetric form q is transferred as $q(m)=\det(m)$ for $m\in M$.

Let $\mathrm{GO}(\mathbb{W})$ denote the symmetric similitude group associated to $\mathbb{W}$. By the definition of $\mathbb{W}$, actually, there exists a morphism of groups $\mathrm{GL}(V_E)\times\mathrm{GL}(V_E)\longrightarrow\mathrm{GO}(\mathbb{W})$. Now, we define a twisted Galois action of $\mathrm{Gal}(E/F)$ on $\mathrm{GL}(V_E)\times\mathrm{GL}(V_E)$ by

$$\mathrm{Gal}(E/F)\times\Big(\mathrm{GL}(V_E)\times\mathrm{GL}(V_E)\Big)\longrightarrow\mathrm{GL}(V_E)\times\mathrm{GL}(V_E);h=(g_1,g_2)\longmapsto{}^\sigma h:=(g_2^\sigma,g_1^\sigma).$$

Write $\overline{\mathrm{GL}(V_E)} = \{h \in \mathrm{GL}(V_E) \times \mathrm{GL}(V_E) | {}^{\sigma}h = h\}$. So there exists an isomorphism of groups $\mathrm{GL}(V_E) \longrightarrow \overline{\mathrm{GL}(V_E)}; g \longmapsto (g, g^{\sigma})$. If given $h \in \mathrm{GL}(V_E) \times \mathrm{GL}(V_E)$, $w \in \mathbb{W} = V_E \otimes_E V_E$, one can verify ${}^{\sigma}h \cdot {}^{\sigma}w = {}^{\sigma}(h \cdot w)$. So it induces a morphism from $\mathrm{GL}(V_E) \simeq \overline{\mathrm{GL}(V_E)}$ to $\mathrm{GO}(M)$. By the fixed basis $\{e_1, e_2\}$, in fact, we obtain a morphism $i : H = \mathrm{GL}_2(E) \longrightarrow \mathrm{GO}(M)$.

Lemma 3.1. *The morphism $i : H = \mathrm{GL}_2(E) \longrightarrow \mathrm{GO}(M)$ is defined by $H \times M \longrightarrow M; (h, m) \longmapsto hm\overline{h}^t$, where $\overline{h}^t$ is the transpose conjugate of h.*

Proof. Let $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H$. By definition, $h \cdot (e_1, e_2) := (e_1, e_2)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ae_1 + ce_2; be_1 + de_2)$. So $(h, 1) \cdot (\alpha e_1 \otimes e_1 + \beta e_1 \otimes e_2 + \gamma e_2 \otimes e_1 + \delta e_2 \otimes e_2) = (a\alpha + b\gamma)e_1 \otimes e_1 + (a\beta + b\delta)e_1 \otimes e_2 + (c\alpha + d\gamma)e_2 \otimes e_1 + (c\beta + d\delta)e_2 \otimes e_2$. Forgetting the basis, we obtain $(h, 1) \cdot \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. Similarly, $(1, \overline{h}) \cdot (\alpha e_1 \otimes e_1 + \beta e_1 \otimes e_2 + \gamma e_2 \otimes e_1 + \delta e_2 \otimes e_2) = (\alpha\overline{a} + \beta\overline{b})e_1 \otimes e_1 + (\beta\overline{d} + \alpha\overline{c})e_1 \otimes e_2 + (\gamma\overline{a} + \delta\overline{b})e_2 \otimes e_1 + (\delta\overline{d} + \gamma\overline{c})e_2 \otimes e_2$, i.e. $(1, \overline{h}) \cdot \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha\overline{a} + \beta\overline{b} & \beta\overline{d} + \alpha\overline{c} \\ \gamma\overline{a} + \delta\overline{b} & \delta\overline{d} + \gamma\overline{c} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\overline{h}^t$. □

Now, we consider the symplectic vector space $V \otimes M$ over F of dimension 8. By the above discussion, there is a map from $G \times H$ to the group $\mathrm{GSp}(V \otimes_F M) \simeq \mathrm{GSp}_8(F)$. Similarly as in section 2, we consider the restriction of the Weil representation $(\rho, \mathrm{GSp}_8(F), W)$ to the group $G \times H$, and will denote it by $(\pi, G \times H, W)$.

Proposition 3.2. *The Weil representation $(\pi, G \times H, W)$ can be realized in the space $W = \mathbb{C}[M \times X_F]$, and the action of $G \times H$ on W is given by the following formulas:*

$$(\pi([h(a), 1])F)(m, \psi) = F(am, \psi), \tag{14}$$

$$(\pi([u(b), 1])F)(m, \psi) = \psi(b \det(m))F(m, \psi), \tag{15}$$

$$(\pi([\omega, 1])F)(m, \psi) = -q^{-2} \sum_{n \in M} F(n, \psi)\psi(B(m, n)), \tag{16}$$

$$(\pi([h'(t), 1])F)(m, \psi) = F(m, \psi^{t^{-1}}), \tag{17}$$

$$(\pi([1, h])F)(m, \psi) = F(h^{-1}mh^{\star -1}\, \mathrm{N}_{E/F}(\det(h)), \psi^{\mathrm{N}_{E/F}(\det(h)^{-1})}), \tag{18}$$

where $h(a), u(b), h'(t) \in G$; $h \in H, m \in M, \psi \in X_F$; $B(m, n) := q(m + n) - q(m) - q(n)$ for $m, n \in M$.

Proof. (14), (15) and (17) follow directly from (1)—(4) in Section 1.
For (18):

$$\Big(\pi([1, h])F\Big)(m, \psi) = \rho\Big(\begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix}\Big)F(m, \psi) = \rho\Big(\begin{pmatrix} 1 & 0 \\ 0 & \mathrm{N}_{E/F}(\det(h)) \end{pmatrix}\Big)\rho\Big(\begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{N}_{E/F}(\det(h))^{-1} \end{pmatrix}\Big)F(m, \psi)$$

$$= F\Big(h^{-1} \cdot \mathrm{N}_{E/F}(\det(h))m, \psi^{\mathrm{N}_{E/F}(\det(h))^{-1}}\Big) = F\Big(h^{-1}mh^{\star -1}\, \mathrm{N}_{E/F}(\det(h)), \psi^{\mathrm{N}_{E/F}(\det(h)^{-1})}\Big).$$

For (16): let $E = F(\xi)$, so $M = \{\begin{pmatrix} x & \alpha \\ \overline{\alpha} & y \end{pmatrix} | x, y \in F, \alpha = a + b\xi \in E\}$. Take $f_1 = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$, $f_2 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}$, $f_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $f_4 = \begin{pmatrix} 0 & \xi \\ \overline{\xi} & 0 \end{pmatrix}$. Suppose $\mathrm{N}_{E/F}(\xi) = a$. Then the set $\{e_1 \otimes f_1, e_1 \otimes f_2, e_1 \otimes f_3, e_1 \otimes f_4; e_2 \otimes f_1, e_2 \otimes f_2, -\frac{1}{2}e_2 \otimes f_3, -\frac{1}{2}a^{-1}e_2 \otimes f_4\}$ is a symplectic basis of $V \otimes_F M$. By such basis, the image of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes 1$ in $\mathrm{GSp}(V \otimes_F M)$ is $\varpi\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix}$ where

$\overline{\omega} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ and $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2a \end{pmatrix}$ with respect to the map $\mathrm{GL}(V) \times GO(M) \longrightarrow \mathrm{GSp}(V \otimes_F M)$. It follows:

$$\begin{aligned}
&\rho(\omega\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix})F(e_2 \otimes m, \psi) \\
&= q^{-2}\sum_{n\in M}\left(\rho\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix}F\right)(e_2 \otimes n, \psi)\psi(\langle e_2 \otimes n, \overline{\omega}^{-1}(e_2 \otimes m)\rangle) \\
&= q^{-2}\chi_q^+(2^2 a)\sum_{n\in M} F(\begin{pmatrix} A^{-1} & 0 \\ 0 & A \end{pmatrix}\cdot e_2 \otimes n, \psi)\psi(\langle e_2 \otimes n, \overline{\omega}^{-1}(e_2 \otimes m)\rangle) \\
&= -q^{-2}\sum_{n\in M} F(e_2 \otimes n, \psi)\psi(\langle\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix}e_2 \otimes n, \overline{\omega}^{-1}(e_2 \otimes m)\rangle) \\
&= -q^{-2}\sum_{n\in M} F(e_2 \otimes n, \psi)\psi(\langle e_2 \otimes n, (\overline{\omega}\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix})^{-1}(e_2 \otimes m)\rangle) \\
&= -q^{-2}\sum_{n\in M} F(e_2 \otimes n, \psi)\psi(\langle e_2 \otimes n, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{-1}(e_2) \otimes m\rangle) \\
&= -q^{-2}\sum_{n\in M} F(e_2 \otimes n, \psi)\psi(\langle e_2 \otimes n, -e_1 \otimes m\rangle) \\
&= -q^{-2}\sum_{n\in M} F(e_2 \otimes n, \psi)\psi(B(m, n)),
\end{aligned}$$

thus we obtain the equality (16). □

3.2. By the standard method to decompose a reducible representation, we are so led to calculate the dimension of the vector space $W[\pi_1]$(see below for its definition).

Let $U = \{x \in E^\times | N_{E/F}(x) = 1\}$, we regard U as a subgroup of H. Let $\mathrm{Irr}_0(H)$ be the set of the isomorphism classes of the irreducible representations π_1 of H, such that π_1 is trivial over U. For each representation $(\pi_1, V_1) \in \mathrm{Irr}_0(H)$, we associate a representation $(\pi_0, W[\pi_1])$ of G, where the vector space $W[\pi_1]$ consists of all the functions: $f : M \times X_F \longrightarrow V_1$ such that

$$f(h^{-1}mh^{\star -1}\, \mathrm{N}_{E/F}(\det(h)), \psi^{\mathrm{N}_{E/F}(\det(h)^{-1})}) = \pi_1(h) \circ f(m, \psi), \tag{19}$$

for $h \in H, (m, \psi) \in M \times X_F$ and the action of G on $W[\pi_1]$ is given by the formulas (14)—(17).

Proposition 3.3. *For the representation $(\pi, G \times H, W)$, we have the following decomposition:*

$$\pi = \bigoplus_{(\pi_1, V_1)\in \mathrm{Irr}_0(H)} W[\pi_1] \otimes V_1.$$

Proof. The representation W has the decomposition $\pi = \oplus_{(\pi_1,V_1)} W_{\pi_1} \otimes V_1$, then $W_{\pi_1} \simeq (W \otimes \check{V}_1)^H \simeq \mathrm{Hom}_H(W, V_1)$. The action of G on $W[\pi_1]$ arises from the definition of π and the above isomorphisms. □

For $(\pi_1, V_1) \in \mathrm{Irr}_0(H)$, we define $W[\pi_1](\xi) = \{f(\xi) | f \in W[\pi_1]\}$ and an H-action on the set $M \times X_F$ as follows:

$$h \cdot (m, \psi) := (hmh^\star\, \mathrm{N}_{E/F}(\det(h))^{-1}, \psi^{\mathrm{N}_{E/F}(\det(h))}) \text{ where } h \in H, \psi \in X_F, m \in M. \tag{20}$$

It is observed that $W[\pi_1](\xi) = V_1^{\mathrm{Stab}_H(\xi)}$ for any $\xi \in M \times X_F$.

Proposition 3.4. *Consider the action of H on $M \times X_F$.*
(1) The distinct orbit of this action can be described as follows:

(i) *Orbit $\{\xi_a\}$, where $\xi_a = (\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \phi^a)$ for any $a \in F^\times$;*

(ii) *Orbit $\{\eta\}$, where $\eta = (\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \phi)$;*

(iii) *Orbit* $\{\delta\}$, *where* $\delta = (\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \phi)$.

(2) The corresponding stabilizer of the canonical element in each orbit is presented as following:

(i) $\mathrm{Stab}_H(\xi_a) = U_2(E)$, *where* $U_2(E) = \{h \in H | hh^\star = 1\}$;

(ii) $\mathrm{Stab}_H(\eta) = H_1$, *where* $H_1 = \{h = \begin{pmatrix} u & b \\ 0 & v \end{pmatrix} | u, v \in U, b \in E\}$;

(iii) $\mathrm{Stab}_H(\delta) = H_2$, *where* $H_2 = \{h \in H | \det(h) \in U\}$.

Proof. We transfer the H-action $\cdot$ to another H-action $\odot$, where $\odot$ is defined by $h \odot (m, \psi) := (hmh^\star, \psi^{\mathrm{N}_{E/F}(\det(h)^{-1})})$. Since the map $\alpha : H \longrightarrow H; h \longmapsto h/\det(h)$ is a group isomorphism and $\alpha(h) \odot (m, \psi) = h \cdot (m, \psi)$, it reduces to consider the action $\odot$.

(1) Every element $m \in M$ corresponds to a hermitian form over a 2-dimension E-vector space V. By the property of hermitian form over finite field and the surjection of the morphism $\mathrm{N}_{E/F} : E^\times \longrightarrow F^\times$, one can find $h \in H$ such that $hmh^\star = \mathrm{diag}(a, b)$ where $(a, b) = (1, 1)$, $(1, 0)$ or $(0, 0)$. Consequently one can calculate the stabilizer $H_{ab} = \mathrm{Stab}_H(\mathrm{diag}(a, b))$ and determine the orbit of H_{ab}-action over X_F to obtain the result.

(2) It is straightforward. □

Lemma 3.5. (i) $U_2(E) = \{\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ -b^q & a^q \end{pmatrix} | u \in U, a, b \in E, \mathrm{N}_{E/F}(a) + \mathrm{N}_{E/F}(b) = 1\}$;

(ii) $|U_2(E)| = (q-1)q(q+1)^2$;

(iii) *If we choose an element* $e_{-1} \in E^\times$, *such that* $e_{-1}^{q+1} = -1$, *then the element* $\begin{pmatrix} 0 & 1 \\ -1 & e_{-1}^q \end{pmatrix} \notin B' U_2(E)$;

(iv) $H = B' U_2(E) \cup B' \begin{pmatrix} 0 & 1 \\ -1 & e_{-1}^q \end{pmatrix} U_2(E)$.

Proof. (i)(ii) follow from [[A], p. 242-243].

(iii)

$$B' U_2(E) = B' S\, U_2(E) = \{\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \cdot \begin{pmatrix} a & b \\ -b^q & a^q \end{pmatrix} | x, z \in E^\times; a, b, y \in E; \mathrm{N}_{E/F}(a) + \mathrm{N}_{E/F}(b) = 1\}$$

$$= \{\begin{pmatrix} xa - yb^q & xb + ya^q \\ -zb^q & za^q \end{pmatrix} | x, z \in E^\times; a, b, y \in E; \mathrm{N}_{E/F}(a) + \mathrm{N}_{E/F}(b) = 1\}.$$

So, $\mathrm{N}_{E/F}(-b^q z) + \mathrm{N}_{E/F}(a^q z) = \mathrm{N}_{E/F}(z) \neq 0$. On the other hand, $\mathrm{N}_{E/F}(-1) + \mathrm{N}_{E/F}(-e_{-1}^q) = 0$, this implies the result.

(iv) It is enough to check that there is an equality about their ranks:

$$|H| = |B' U_2(E)| + |B' \begin{pmatrix} 0 & 1 \\ -1 & e_{-1}^q \end{pmatrix} U_2(E)|.$$

By calculation, $B' \cap U_2(E) = \{\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^q \end{pmatrix} | u, a \in U\}$. So

$$|B' U_2(E)| = |B'| |\frac{U_2(E)}{B' \cap U_2(E)}| = |B'| \cdot (q-1)q.$$

Now let $g = \begin{pmatrix} ua & ub \\ -b^q & a^q \end{pmatrix} \in U_2(E)$, $g_0 = \{\begin{pmatrix} 0 & 1 \\ -1 & e_{-1}^q \end{pmatrix}$. Then $g_0^{-1} = \{\begin{pmatrix} e_{-1}^q & -1 \\ 1 & 0 \end{pmatrix}$, and $g_0 g g_0^{-1} = \begin{pmatrix} (a - be_{-1})^q & b^q \\ e_{-1}^q[(a - e_{-1}b)^q - u(a - e_{-1}b)] & ua + (be_{-1})^q \end{pmatrix}$ which is an element in B' iff $(a - be_{-1})^{q-1} = u$.

Hence

$$B' \cap g_0 U_2(E) g_0^{-1} = \{\begin{pmatrix} (a - be_{-1})^q & b^q \\ 0 & (a - be_{-1})^{-1} \end{pmatrix} | \mathrm{N}_{E/F}(a) + \mathrm{N}_{E/F}(b) = 1\}.$$

If $\mathrm{N}_{E/F}(a) + \mathrm{N}_{E/F}(b) = 1$, then $a - be_{-1} \neq 0$. So $|B' \cap g_0 U_2(E) g_0^{-1}| = |S\, U_2(E)|$. From this, we obtain

$$|g_0 U_2(E) g_0^{-1} / B' \cap g_0 U_2(E) g_0^{-1}| = q + 1.$$

So

$$|B' g_0 U_2(E)| = |B'| |g_0 U_2(E) g_0^{-1} / B' \cap g_0 U_2(E) g_0^{-1}| = |B'| \cdot (q+1).$$

Finally

$$|B'U_2(E)| + |B'g_0U_2(E)| = |B'| \cdot (q-1)q + |B'| \cdot (q+1) = |B'| \cdot (q^2+1) = |H|.$$

□

Let's determine the dimension of the vector space $W[\pi_1]$ for each $\pi_1 \in \mathrm{Irr}_0(H)$.

Lemma 3.6. *Let (π_1, V_1) be an irreducible representation of H, which belongs to $\mathrm{Irr}_0(H)$.*
(1) For $\pi_1 = \Psi \cdot 1_H$ where $\Psi \in \mathrm{Irr}(E^\times)$:

(a) *if $\Psi \neq \Psi^q$ then $W[\pi_1] = 0$.*
(b) *if $\Psi = \Psi^q$ then $\dim_{\mathbb{C}} W[\pi_1](\xi_a) = \dim_{\mathbb{C}} W[\pi_1](\eta) = \dim_{\mathbb{C}} W[\pi_1](\delta) = 1$.*

(2) For $\pi_1 = \Psi \cdot \mathrm{St}_H$ where $\Psi \in \mathrm{Irr}(E^\times)$:

(a) *if $\Psi \neq \Psi^q$ then $W[\pi_1] = 0$.*
(b) *if $\Psi = \Psi^q$ then $\dim_{\mathbb{C}} W[\pi_1](\xi_a) = \dim_{\mathbb{C}} W[\pi_1](\eta) = 1$ and $\dim_{\mathbb{C}} W[\Psi \cdot \mathrm{St}_H](\delta) = 0$.*

(3) For $\pi_1 = \Pi_{\Lambda,\Sigma}$ where $\Lambda \neq \Sigma \in \mathrm{Irr}(E^\times)$:

(a) *if $\Lambda = \Lambda^q, \Sigma = \Sigma^q$ then $\dim_{\mathbb{C}} W[\Pi_{\Lambda,\Sigma}](\xi_a) = 1$, $\dim_{\mathbb{C}} W[\Pi_{\Lambda,\Sigma}](\eta) = 2$, $\dim_{\mathbb{C}} W[\Pi_{\Lambda,\Sigma}](\delta) = 0$.*
(b) *if $\Lambda = \Sigma^q, \Sigma = \Lambda^q$ then $\dim_{\mathbb{C}} W[\Pi_{\Lambda,\Sigma}](\xi_a) = 1$, $\dim_{\mathbb{C}} W[\Pi_{\Lambda,\Sigma}](\eta) = \dim_{\mathbb{C}} W[\Pi_{\Lambda,\Sigma}](\delta) = 0$.*
(c) *for the other kind of $\Lambda \neq \Sigma \in \mathrm{Irr}(E^\times)$, $W[\Pi_{\Lambda,\Sigma}] = 0$.*

(4) For $\pi_1 = \Pi_\Theta$ where $\Theta \in \mathrm{Irr}(E_1^\times) - \mathrm{Irr}(E^\times)$ for some quadratic extension E_1 of E:

(a) *$W[\Pi_\Theta] = 0$.*

Proof. (1) It is easily checked that the image of the map $\det : \mathrm{Stab}_H(\xi) \longrightarrow E^\times$ is U when $\xi = \xi_a, \eta, \delta$. It implies that $V_1 = V_1^{\mathrm{Stab}_H(\xi)}$ iff Ψ is trivial over U.
(2) Let us denote by $(\pi_2, V_2) = \mathrm{Ind}_{B'}^H(\Psi \cdot 1_{B'})$. Firstly $W[\pi_2](\xi) = W[\Psi \cdot 1_H](\xi) \oplus W[\Psi \cdot \mathrm{St}_H](\xi)$ for $\xi = \xi_a, \eta, \delta$. We have already determined the set $W[\Psi \cdot 1_H](\xi)$ in (1), so it remains to calculate the dimension $W[\pi_2](\xi)$ for $\xi = \xi_a, \eta, \delta$.

For $\xi = \xi_a$, $\mathrm{Stab}_H(\xi) = U_2(E)$ and $H = B'U_2(E) \cup B'\begin{pmatrix} 0 & 1 \\ -1 & e_{-1}^q \end{pmatrix} U_2(E)$, for a fixed $e_{-1} \in E^\times$ such that $\mathrm{N}_{E/F}(e_{-1}) = -1$. So $V_2^{U_2(E)}$ is generated by the functions α, β in V_2, where $supp(\alpha) = B'U_2(E)$, $\alpha(bu) := \Psi \cdot 1_{B'}(b)$ for $b \in B', u \in U_2(E)$ and $\Psi \cdot 1_{B'}$ is trivial over $B' \cap U_2(E)$; $supp(\beta) = B'\begin{pmatrix} 0 & 1 \\ -1 & e_{-1}^q \end{pmatrix} U_2(E)$, $\beta\big(b\begin{pmatrix} 0 & 1 \\ -1 & e_{-1}^q \end{pmatrix} u\big) := \Psi \cdot 1_{B'}(b)$ and $\Psi \cdot 1_{B'}$ is trivial over $B' \cap \begin{pmatrix} 0 & 1 \\ -1 & e_{-1}^q \end{pmatrix} U_2(E) \begin{pmatrix} 0 & 1 \\ -1 & e_{-1}^q \end{pmatrix}^{-1}$. By Lemma 3.5, $B' \cap U_2(E) = \{\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & a^q \end{pmatrix} | u, a \in U\}$, then $\alpha \neq 0$ iff $\Psi = \Psi^q$. On the other hand, by Lemma 3.5, $\begin{pmatrix} 0 & 1 \\ -1 & e_{-1}^q \end{pmatrix} g \begin{pmatrix} e_{-1}^q & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -(be_{-1})^q + a^q & b^q \\ u(-b - ae_{-1}^q) + (ae_{-1})^q - (be_{-1}^2)^q & ua + (be_{-1})^q \end{pmatrix}$ which implies $\begin{pmatrix} 0 & 1 \\ -1 & e_{-1}^q \end{pmatrix} g \begin{pmatrix} e_{-1}^q & -1 \\ 1 & 0 \end{pmatrix} \in B'$ iff $u(ae_{-1}^q + b) = (ae_{-1})^q - (be_{-1}^2)^q$ and $\mathrm{N}_{E/F}(a) + \mathrm{N}_{E/F}(b) = 1, u \in U$. In particular, if $a = 0$, then the condition is $u = (e_{-1}b)^{q-1}, \mathrm{N}_{E/F}(e_{-1}b) = -1$; if $b = 0$, then the condition is $u = a^{q-1}, \mathrm{N}_{E/F}(a) = 1$. By calculation, the set $\{u | u = a^{q-1}$ with $a \in E^\times$ and $\mathrm{N}_{E/F}(a) = \pm 1\}$ is equal to U. Hence $\{\det(g) = u | \begin{pmatrix} 0 & 1 \\ -1 & e_{-1}^q \end{pmatrix} g \begin{pmatrix} e_{-1}^q & -1 \\ 1 & 0 \end{pmatrix} \in B', g = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ -b^q & a^q \end{pmatrix} \in U_2(E)\} = U$. Finally, we see $\beta \neq 0$ iff $\Psi = \Psi^q$.

For $\xi = \eta$, $\mathrm{Stab}_H(\xi) = H_1 = N' \rtimes T''$, where $T'' = \{\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} | u, v \in U\}$, therefore $V_2^{H_1} = (V_2^{N'})^{T''}$. By Lemma 2.4, the vector space $V_2^{N'}$ is generated by functions $f_{\Psi,\Psi}, g_{\Psi,\Psi}$ and for $t \in T$, $t \cdot f_{\Psi,\Psi} = \Psi \otimes \Psi(t) f_{\Psi,\Psi}$, $t \cdot g_{\Psi,\Psi} = \Psi \otimes \Psi(t) g_{\Psi,\Psi}$. So we obtain: if $\Psi = \Psi^q$, $f_{\Psi,\Psi}, g_{\Psi,\Psi} \in V_2^{H_1}$; if $\Psi \neq \Psi^q$, $V_2^{H_1} = 0$.

For $\xi = \delta$, $\mathrm{Stab}_H(\delta) = H_2$, $H = B'H_2$ and $B' \cap H_2 = \{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} | ad \in U\}$. So $\dim_{\mathbb{C}} V_2^{H_2} = 1$(resp. 0) if Ψ is trivial (resp. not trivial) over U.
(3) For $\xi = \xi_a$, $V_1^{U_2(E)}$ is generated by two functions α, β in V_1, where $supp(\alpha) = B'U_2(E)$, $\alpha(bu) := \Lambda\Sigma(b)$ for

$b \in B', u \in U_2(E)$ and $\Lambda\otimes\Sigma$ is trivial over $B'\cap U_2(E)$; $supp(\beta) = B'\begin{pmatrix} 0 & 1 \\ -1 & e_{-1}^q\end{pmatrix}U_2(E), \beta\Big(b\begin{pmatrix} 0 & 1 \\ -1 & e_{-1}^q\end{pmatrix}u\Big) := \Lambda\otimes\Sigma(b)$ and $\Lambda\otimes\Sigma$ is trivial over $B'\cap\begin{pmatrix} 0 & 1 \\ -1 & e_{-1}^q\end{pmatrix}U_2(E)\begin{pmatrix} e_{-1}^q & -1 \\ 1 & 0\end{pmatrix}$. By Lemma 3.5, $B'\cap U_2(E) = \{\begin{pmatrix} ua & 0 \\ 0 & a^q\end{pmatrix}|a, u \in U\}$; Therefore $\alpha \neq 0$ iff $\Lambda = \Lambda^q$ and $\Sigma = \Sigma^q$. On the other hand, by Lemma 3.5, $B'\cap\begin{pmatrix} 0 & 1 \\ -1 & e_{-1}^q\end{pmatrix}U_2(E)\begin{pmatrix} e_{-1}^q & -1 \\ 1 & 0\end{pmatrix} = \{\begin{pmatrix} (a-be_{-1})^q & b^q \\ 0 & (a-be_{-1})^{-1}\end{pmatrix}|\,\mathrm{N}_{E/F}(a)+\mathrm{N}_{E/F}(b) = 1\}$. Set $t = a - be_{-1}$, $s = a + be_{-1}$, then $\mathrm{N}_{E/F}(a)+\mathrm{N}_{E/F}(b) = 1$ is equivalent to $ts^q + st^q = 2$. And $\beta \neq 0$ iff $\beta\Big(\begin{pmatrix} t^q & b^q \\ 0 & t^{-1}\end{pmatrix}\Big) = \Lambda^q\Sigma^{-1}(t) = 1$ for $ts^q + st^q = 2$. By considering $t = s^{-q}$, we obtain $\beta \neq 0$ iff $\Lambda^q\Sigma^{-1} = 1$, obviously $\Lambda \neq \Lambda^q$, hence $\beta \neq 0$ iff $\Sigma = \Lambda^q$, and $\Sigma \neq \Lambda$.

For $\xi = \eta$, $\mathrm{Stab}_H(\xi) = N' \rtimes T''$, and $V_1^{N'}$ is generated by functions $f_{\Lambda,\Sigma}, g_{\Lambda,\Sigma}$ by Lemma 2.4. By considering the T''-action on $V_1^{N'}$, we obtain: if $\Lambda = \Lambda^q, \Sigma = \Sigma^q$ then $f_{\Lambda,\Sigma}, g_{\Lambda,\Sigma} \in V_1^{H_1}$; in other cases $V_1^{H_1} = 0$.

For $\xi = \delta$, $\mathrm{Stab}_H(\xi) = H_2$ and $H = B'H_2$, $B'\cap H_2 = \{\begin{pmatrix} a & b \\ 0 & d\end{pmatrix}|ad \in U\}$, then $V_1^{H_2}$ is generated by the function f where $f(bh) = (\Lambda\otimes\Sigma)(b)$ for $b \in B', h \in H_2$ and $\Lambda\otimes\Sigma$ is trivial over $B'\cap H_2$, this implies $V_1^{H_2} = 0$.

(4) For $\xi = \xi_a$, $U_2(E) \supseteq SU_2(E) = SL_2(E)\cap U_2(E)$ and there exists $h \in H$ such that $hSU_2(E)h^{-1} = SL_2(F)$ (see [A], p. 242). Hence $V_1^{U_2(E)} \subseteq V_1^{SU_2(E)} \simeq V_1^{SL_2(F)} = 0$.

For $\xi = \eta, \delta$, $H_i \supseteq N'$ and $V_1^{N'} = 0$. So $V_1^{H_i} = 0$ for $i = 1, 2$. □

Corollary 3.7. *Let π_1 be an irreducible representation of H in $\mathrm{Irr}_0(H)$. Then:*
(i) $\dim_{\mathbb{C}} W[\Psi\cdot 1_H] = q+1$ *if* $\Psi = \Psi^q$, $\Psi \in \mathrm{Irr}(E^\times)$;
(ii) $\dim_{\mathbb{C}} W[\Psi\cdot \mathrm{St}_H] = q$ *if* $\Psi = \Psi^q$, $\Psi \in \mathrm{Irr}(E^\times)$;
(iii) $\dim_{\mathbb{C}} W[\Pi_{\Lambda,\Sigma}] = q+1$ *if* $\Lambda, \Sigma \in \mathrm{Irr}(E^\times)$ *and* $\Lambda \neq \Sigma, \Lambda = \Lambda^q, \Sigma = \Sigma^q$;
(iv) $\dim_{\mathbb{C}} W[\Pi_{\Lambda,\Sigma}] = q-1$ *if* $\Lambda, \Sigma \in \mathrm{Irr}(E^\times)$, *and* $\Lambda \neq \Sigma, \Lambda = \Sigma^q, \Sigma = \Lambda^q$;
the above list are all representations $\pi_1 \in \mathrm{Irr}_0(H)$, such that $W[\pi_1] \neq 0$.

3.3. **The representation** $(\pi_0, W[\pi_1])$ **I.** In this Subsection, let $\pi_1 = \Psi\cdot 1_H$ where $\Psi = \Psi^q \in \mathrm{Irr}(E^\times)$ and $\Psi = \psi\circ \mathrm{N}_{E/F}$ for some $\psi \in \mathrm{Irr}(F^\times)$.

The vector space $W[\pi_1]$ is generated by functions $\{F_a, R, S : M\times X_F \longrightarrow V_1$ for any $a \in F^\times\}$. Namely they all satisfy the equality (19), and $\mathrm{supp}(F_a)$= Orbit$\{\xi_a\}$, $F_a(\xi_a) = v_0 \in V_1^{U_2(E)}$ for any $a \in F^\times$; $\mathrm{supp}(R)$= Orbit$\{\eta\}$, $R(\eta) = v_1 \in V_1^{H_1}$; $\mathrm{supp}(S)$= Orbit$\{\delta\}$, $S(\delta) = v_2 \in V_1^{H_2}$.

Since for the representation π_0, the action of B on $W[\pi_1]$ is linear, it is easy to verify the following formulas:
(I) $\pi_0(h(r))F_{ar^2} = \psi(r^2)F_a$; (II) $\pi_0(h'(t))F_{at^{-1}} = F_a$; (III) $\pi_0(u(b))F_a = \phi^a(b)F_a$; (IV) $\pi_0(h(r))R = R$; (V) $\pi_0(h'(t))R = \psi(t)R$;
(VI) $\pi_0(u(b))R = R$; (VII) $\pi_0(h(r))S = S$; (VIII) $\pi_0(h'(t))S = \psi(t)S$; (IX) $\pi_0(u(b))S = S$; (X) $\pi_0(\omega)S \neq S$.

By the above formulas (I)—(IX), we could calculate the characters:
$\chi_{\pi_0}(\begin{pmatrix} r & 0 \\ 0 & r\end{pmatrix}) = (q+1)\psi(r^2)$, $\chi_{\pi_0}(\begin{pmatrix} r_1 & 0 \\ 0 & r_2\end{pmatrix}) = 2\psi(r_1r_2)$, and $\chi_{\pi_0}(\begin{pmatrix} r & 1 \\ 0 & r\end{pmatrix}) = \psi(r^2)$ for $t_1, t_2, r, r_1 \neq r_2 \in F^\times$. So the restriction of π_0 and $\psi\cdot\mathrm{Ind}_B^G 1_G$ on B coincides with each other, i.e. $\mathrm{Res}_B^G(\pi_0) \simeq 2\sigma_{\psi,\psi}\oplus\psi^2\otimes\sigma$ and the isotypic component $2\sigma_{\psi,\psi}$ is spanned by functions $\{R, S\}$. By Proposition 1.3, $\pi_0 \simeq \psi\cdot\mathrm{Ind}_B^G 1_G\cdots(1)$ or $\pi_0 \simeq 2\psi\cdot 1_G\oplus\pi_\theta$ for certain regular character θ of $E^\times\cdots(2)$. But the above formula (X) $\pi_0(\omega)S \neq S$ means that π_0 has only one isotypic component $\psi\cdot 1_G$, hence the above case (2) is impossible. Finally we could obtain $\pi_0 \simeq \psi\cdot\mathrm{Ind}_B^G 1_G$.

3.4. **The representation** $(\pi_0, W[\pi_1])$ **II.** In this Subsection, let $\pi_1 = \Psi\cdot\mathrm{St}_H$, where $\Psi = \Psi^q \in \mathrm{Irr}(E^\times)$, $\Psi = \psi\cdot\mathrm{N}_{E/F}$ for some $\psi \in \mathrm{Irr}(F^\times)$.

The vector space $W[\pi_1]$ is generated by functions $\{F_a, R : M\times X_F \longrightarrow V_1$ for any $a \in F^\times\}$. They all satisfy the equality (19), also $\mathrm{supp}(F_a)$= Orbit$\{\xi_a\}$, $F_a(\xi_a) = v_0 \in V_1^{U_2(E)}$ for any $a \in F^\times$; $\mathrm{supp}(R)$= Orbit$\{\eta\}$,

$R(\eta) = v_1 = q^2 f_{\Psi,\Psi} - g_{\Psi,\Psi} \in V_1^{H_1}$ by Lemma 2.4.

By (14)— (17), we can obtain:
(XI) $\pi_0(h(r))F_{ar^2} = \psi(r^2)F_a$; (XII) $\pi_0(h'(t))F_{at^{-1}} = F_a$; (XIII) $\pi_0(u(b))F_a = \phi^a(b)F_a$; (XIV) $\pi_0(h(r))R = R$; (XV) $\pi_0(h'(t))R = \psi(t)R$; (XVI) $\pi_0(u(b))R = R$;

Lemma 3.8. *Let $M^{(1)} = \{n \in M | \operatorname{rank} n = 1\}$. Then:*

(a) $M^{(1)} = \{\begin{pmatrix} s & 0\\ 0 & 0\end{pmatrix} | s \in F^\times\} \cup \{s\begin{pmatrix} \mathrm{N}_{E/F}(b) & b\\ b^q & 1\end{pmatrix} | s \in F^\times, b \in E\}$.

(b) $s\begin{pmatrix} \mathrm{N}_{E/F}(b) & b\\ b^q & 1\end{pmatrix} = u(b)\begin{pmatrix} 0 & 0\\ 0 & s\end{pmatrix}u(b)^\star$.

Proof. See [[A], p. 247]. □

$$(\pi_0(\omega)R)(\eta) = -q^{-2}\sum_{n\in M} R(n,\phi)\phi\Big(B\Big(\begin{pmatrix} 1 & 0\\ 0 & 0\end{pmatrix}, n\Big)\Big)$$
$$\overset{\mathrm{supp}(R)=M^{(1)}}{=} -q^{-2}\sum_{n\in M^{(1)}} R(n,\phi)\phi\Big(B\Big(\begin{pmatrix} 1 & 0\\ 0 & 0\end{pmatrix}, n\Big)\Big)$$
$$= -q^{-2}\sum_{s\in F^\times}\Big[R\Big(\begin{pmatrix} s & 0\\ 0 & 0\end{pmatrix},\phi\Big)\phi\Big(B\Big(\begin{pmatrix} 1 & 0\\ 0 & 0\end{pmatrix},\begin{pmatrix} s & 0\\ 0 & 0\end{pmatrix}\Big)\Big) + \sum_{b\in E} R\Big(u(b)\begin{pmatrix} 0 & 0\\ 0 & s\end{pmatrix}u(b)^\star,\phi\Big)\phi\Big(B\Big(\begin{pmatrix} 1 & 0\\ 0 & 0\end{pmatrix}, u(b)\begin{pmatrix} 0 & 0\\ 0 & s\end{pmatrix}u(b)^\star\Big)\Big)\Big]$$
$$= -q^{-2}\sum_{s\in F^\times}[\pi_0\big(h(s)\big)R(\eta) + \sum_{b\in E}\pi_1\Big(\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}u(-b)\Big)R\Big(\begin{pmatrix} s & 0\\ 0 & 0\end{pmatrix},\phi\Big)\phi(s)]$$
$$= -q^{-2}\sum_{s\in F^\times}[v_1 + \sum_{b\in E}\pi_1\Big(\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}u(-b)\Big)R\Big(\begin{pmatrix} s & 0\\ 0 & 0\end{pmatrix},\phi\Big)\phi(s)]$$
$$= -q^{-2}\sum_{s\in F^\times}[v_1 + \phi(s)\sum_{b\in E}\pi_1\Big(\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}\Big)v_1]$$
$$= -q^{-2}[(q-1)v_1 - q^2\pi_1\Big(\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}\Big)v_1] \neq R(\eta).$$

It follows: (XVII) $\pi_0(\omega)R \neq R$.

By the formulas (XI)—(XVI), we obtain $\mathrm{Res}^G_B(\pi_0) = \mathrm{Res}^G_B(\psi\cdot \mathrm{St}_G)$. Consequently by the formula (XVII), π_0 has no $\psi\cdot 1_G$ isotypic component. Hence by comparing this with the result in Proposition 1.3, we obtain $\pi_0 \simeq \psi\cdot\mathrm{St}_G$.

3.5. **The representation** $(\pi_0, W[\pi_1])$ **III.** In this Subsection, let $\pi_1 = \Pi_{\Lambda,\Sigma}$ for $\Lambda \neq \Sigma$ and $\Lambda = \lambda\circ \mathrm{N}_{E/F}, \Sigma = \sigma\circ\mathrm{N}_{E/F} \in \mathrm{Irr}(E^\times)$.

The vector space $V_1^{H_1}(= V_1^{N'})$ is generated by functions $\{f_{\Lambda,\Sigma}, g_{\Lambda,\Sigma}\}$ by Lemma 2.4. Let $\Delta: M\times X_F \longrightarrow V_1$ such that it satisfies (19) and supp(Δ)= Orbit$\{\eta\}$, $\Delta(\eta) = f_{\Lambda,\Sigma}$. Then:

$$(i)\qquad \big(\pi_0(h(r))(\Delta)\big)(\eta) = \Delta\Big(\begin{pmatrix} r & 0\\ 0 & 0\end{pmatrix},\phi\Big) = \Delta\Big(\begin{pmatrix} x^{-1} & 0\\ 0 & x\end{pmatrix}^{-1}\begin{pmatrix} 1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix} x^{-q} & 0\\ 0 & x^q\end{pmatrix}^{-1},\phi\Big) = \Pi_{\Lambda,\Sigma}\Big(\begin{pmatrix} x^{-1} & 0\\ 0 & x\end{pmatrix}\Big)\Delta(\eta)$$
$$= \Pi_{\Lambda,\Sigma}\Big(\begin{pmatrix} x^{-1} & 0\\ 0 & x\end{pmatrix}\Big)[f_{\Lambda,\Sigma}] = \Lambda(x^{-1})\Sigma(x)f_{\Lambda,\Sigma} = \lambda(r^{-1})\sigma(r)f_{\Lambda,\Sigma} \text{ for } \mathrm{N}_{E/F}(x) = r.$$

$$(ii)\qquad \big(\pi_0(h'(t))(\Delta)\big)(\eta) = \Delta\Big(\begin{pmatrix} 1 & 0\\ 0 & 0\end{pmatrix},\phi^{t^{-1}}\Big) = \Delta\Big(\mathrm{N}_{E/F}(x)\begin{pmatrix} x & 0\\ 0 & 1\end{pmatrix}^{-1}\begin{pmatrix} 1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix} x^q & 0\\ 0 & 1\end{pmatrix}^{-1},\phi^{\mathrm{N}_{E/F}(x)^{-1}}\Big)$$
$$= \Pi_{\Lambda,\Sigma}\Big(\begin{pmatrix} x & 0\\ 0 & 1\end{pmatrix}\Big)[\Delta(\eta)] = \Pi_{\Lambda,\Sigma}\Big(\begin{pmatrix} x & 0\\ 0 & 1\end{pmatrix}\Big)[f_{\Lambda,\Sigma}] = \Lambda(x)f_{\Lambda,\Sigma} = \lambda(t)f_{\Lambda,\Sigma} = \lambda(t)\Delta(\eta) \text{ for } \mathrm{N}_{E/F}(x) = r.$$

By the above (i) and (ii), $\pi_0(h(r))\Delta = \lambda(r^{-1})\sigma(r)\Delta$, $\pi_0(h'(t))\Delta = \lambda(t)\Delta$. In particular, we obtain $\pi_0\Big(\begin{pmatrix} t_1 & 0\\ 0 & t_2\end{pmatrix}\Big)\Delta = \lambda(t_2)\sigma(t_1)\Delta$. It follows that $\mathrm{Hom}_G(\pi_0, \mathrm{Ind}_B^G \lambda\otimes\sigma) \simeq \mathrm{Hom}_G(\pi_0, \mathrm{Ind}_B^G \sigma\otimes\lambda) \neq 0$. Since $\dim_{\mathbb{C}} \pi_0 = q+1$, surely $\pi_0 \simeq \pi_{\lambda,\sigma}$.

3.6. **The representation** $(\pi_0, W[\pi_1])$ **IV.** In this Subsection, let $\pi_1 = \Pi_{\Lambda,\Sigma}$ where $\Lambda \neq \Sigma \in \mathrm{Irr}(E^\times)$ and $\Lambda = \Sigma^q, \Sigma = \Lambda^q$.

We start with recalling some explicit models for certain representations[cf. [A], p.21, Definition 2 and p.53, Proposition 4]:

Model for $\pi_1 = \Pi_{\Lambda,\Sigma}$: π_1 could be realized in the vector space V_1 where V_1 is spanned by functions $v : E^2\times E^\times \longrightarrow \mathbb{C}$ such that $v(a(e_1,e_2); a^{-1}b^{-1}e_3) = \Lambda(a)\Sigma(b)v(e_1,e_2;e_3)\cdots(\star)$ and $(\pi(h)v)(e_1,e_2;e_3) = v((e_1,e_2)h; e_3\det(h)^{-1})$ for $e_1,e_2\in E; a,b,e_3\in E^\times, h\in H$.

Model for the cuspidal representation π_Λ of G, associated to the regular character Λ of $E^\times$: π_Λ could be realized in the vector space $\mathbb{C}[X_F]$, where the actions are following:

(1) $\pi_\Lambda\Big(\begin{pmatrix} r & 0\\ 0 & r\end{pmatrix}\Big)f = \Lambda(r)f$;

(2) $\Big(\pi_\Lambda\begin{pmatrix} 1 & 0\\ 0 & t\end{pmatrix} f\Big)(\psi) = f(\psi^{t^{-1}})$;

(3) $\Big(\pi_\Lambda\begin{pmatrix} 1 & s\\ 0 & 1\end{pmatrix} f\Big)(\psi) = \psi(s)f(\psi)$;

(4) $\Big((\pi_\Lambda(\omega))f\Big)(\psi) = -q^{-1}\sum_{y\in E^\times}\psi(\mathrm{Tr}_{E/F}(y))\Lambda(y)f(\psi^{N_{E/F}(y)})$ where $\psi\in X_F$, $t,r\in F^\times$, $s\in F$.

Now we concentrate on the representation $(\pi_0, G, W[\pi_1])$. The vector space $W[\pi_1]$ is generated by functions $F_a : M\times X_F \longrightarrow V_1$, it satisfies (19) and $supp(F_a)$= Orbit$\{\xi_a\}$, $F_a(\xi_a) = v_1 \in V_1^{U_2(E)}$ for any $a\in F^\times$. Using the above model, we choose one element v_1 as follows:

$v_1 : E^2\times E^\times \longrightarrow \mathbb{C}$, it satisfies above $(\star)$ and $supp(v_1) = \cup_{u\in U}$ Orbit $\{(1, ue_{-1}; 1)\}$, $v_1(1, ue_{-1}; 1) = \Lambda(u)$ where Orbit$\{(1,ue_{-1};1)\}$=$\{(a, ue_{-1}a; a^{-1}b^{-1}) \in E^2\times E^\times | a,b\in E^\times\}$, e_{-1} is a fixed element in $E^\times$ such that $\mathrm{N}_{E/F}(e_{-1}) = -1$.

Now we check that $v_1 \in V_1^{U_2(E)}$:

For $g = \begin{pmatrix} u & 0\\ 0 & 1\end{pmatrix}\in U_2(E)$, supp$(g\cdot v_1)$=supp$(v_1)$, and $gv_1(1, u_0e_{-1}; 1) = v_1(u, u_0e_1; u^{-1}) = \Lambda(u)v_1(1, u_0u^{-1}e_{-1}; 1) = \Lambda(u)\Lambda(u_0u^{-1}) = \Lambda(u_0) = v_1(1, u_0e_{-1}; 1)$. Hence in this case, $gv_1 = v_1$.

For $g = \begin{pmatrix} a & b\\ -b^q & a^q\end{pmatrix}\in U_2(E)$, $gv_1(1, ue_{-1}; 1) = v_1(a - b^que_{-1}, b + ue_{-1}a^q; 1) = v_1(a - b^que_{-1}, ue_{-1}(a-b^que_{-1})^q; 1)$ $= \Lambda(a-b^que_{-1})v_1(1, ue_{-1}(a-b^que_{-1})^{q-1}; a - b^que_{-1}) = \Lambda(a-b^que_{-1})\Lambda(u(a-b^que_{-1})^{q-1})\Sigma^{-1}(a-b^que_{-1})$ $= \Lambda(u) = v_1(1, ue_{-1}; 1)$. We also have supp$(gv_1)$=supp$(v_1)$. Therefore, we obtain $gv_1 = v_1$ in this case.

We define an intertwining operator between π_Λ and π_0 by

$$j : \pi_\Lambda \longrightarrow W[\pi_1]; f\longmapsto j(f) = \sum_{a\in F^\times} f(\phi^a)F_a, \text{ i.e. } j(f)(\xi_a) = f(\phi^a)v_1.$$

Claim: $j(\pi_\Lambda(g)f) = \pi_0(g)j(f)$ for $g\in G$.

Proof: (1) $g = \begin{pmatrix} x & y\\ 0 & z\end{pmatrix}\in B$.

$$\pi_0\Big(\begin{pmatrix} x & y\\ 0 & z\end{pmatrix}\Big)j(f)(\xi_a) = \sum_{t\in F^\times} f(\phi^t)\pi_0\Big(\begin{pmatrix} x & y\\ 0 & z\end{pmatrix}\Big)F_t(\xi_a)$$
$$= \sum_{t\in F^\times} f(\phi^t)\pi_0\Big(\begin{pmatrix} x & 0\\ 0 & x^{-1}\end{pmatrix}\begin{pmatrix} 1 & 0\\ 0 & xz\end{pmatrix}\begin{pmatrix} 1 & x^{-1}y\\ 0 & 1\end{pmatrix}\Big)F_t(\xi_a)$$
$$= \sum_{t\in F^\times} f(\phi^t)F_t\Big(\begin{pmatrix} x & 0\\ 0 & x\end{pmatrix}, \phi^{ax^{-1}z^{-1}}\Big)\phi^a(yz^{-1})$$

$$\overset{\mathrm{N}_{E/F}(r)=x}{=} \sum_{t\in F^\times} f(\phi^t)F_t\Big(\begin{pmatrix} r^{-1} & 0\\ 0 & r^{-1}\end{pmatrix}\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix} r^{-q} & 0\\ 0 & r^{-q}\end{pmatrix}\mathrm{N}_{E/F}(r^2), \phi^{\mathrm{N}_{E/F}(r^{-2})axz^{-1}}\Big)\phi^a(yz^{-1})$$

$$= \sum_{t\in F^\times} f(\phi^t)\pi_1\Big(\begin{pmatrix} r & 0\\ 0 & r\end{pmatrix}\Big)F_t\Big(\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}, \phi^{axz^{-1}}\Big)\phi^a(yz^{-1})$$

$$= f(\phi^{axz^{-1}})\pi_1\Big(\begin{pmatrix} r & 0\\ 0 & r\end{pmatrix}\Big)v_1\phi^a(yz^{-1})$$

$$= f(\phi^{axz^{-1}})\Lambda(r)\Sigma(r)v_1\phi^a(yz^{-1}) = f(\phi^{axz^{-1}})\Lambda(x)v_1\phi^a(yz^{-1}).$$

So $\pi_0\begin{pmatrix} x & y\\ 0 & z\end{pmatrix} j(f) = \sum_{a\in F^\times} f(\phi^{ax^{-1}z})\Lambda(x)\phi^a(yz^{-1})F_a$.

On the other hand,

$$j(\pi_\Lambda\begin{pmatrix} x & y\\ 0 & z\end{pmatrix} f) = \sum_{a\in F^\times}[\pi_\Lambda\begin{pmatrix} x & y\\ 0 & z\end{pmatrix} f](\phi^a)F_a$$

$$= \sum_{a\in F^\times}[\pi_\Lambda\Big(\begin{pmatrix} x & 0\\ 0 & x\end{pmatrix}\begin{pmatrix} 1 & 0\\ 0 & x^{-1}z\end{pmatrix}\begin{pmatrix} 1 & x^{-1}y\\ 0 & 1\end{pmatrix}\Big) f](\phi^a)F_a$$

$$= \sum_{a\in F^\times}\Lambda(x)f(\phi^{axz^{-1}})\phi^{axz^{-1}}(x^{-1}y)F_a$$

$$= \sum_{a\in F^\times}\Lambda(x)f(\phi^{axz^{-1}})\phi^a(yz^{-1})F_a.$$

(2) $g = \omega$.

$$\Big(\pi_0(\omega)j(f)\Big)(\xi_a) = -q^{-2}\sum_{n\in M}\phi^a(B(\mathrm{Id}_H, n))j(f)(n,\phi^a)$$

$$\overset{\text{consider} supp\left(j(f)\right)}{=} -q^{-2}\sum_{n\in M, \det n\neq 0}\phi^a(B(\mathrm{Id}_H, n))j(f)(n,\phi^a)$$

$$= -q^{-2}|U_2(E)|^{-1}\sum_{h\in H}\phi^a\Big(B(\mathrm{Id}_H, \mathrm{N}_{E/F}(\det(h))h^{-1}h^{\star-1})\Big)j(f)(\mathrm{N}_{E/F}(\det(h))h^{-1}h^{\star-1},\phi^a)$$

$$= -q^{-2}|U_2(E)|^{-1}\sum_{h\in H}\phi^a\Big(B(\mathrm{Id}_H, \mathrm{N}_{E/F}(\det(h))h^{-1}h^{\star-1})\Big)f(\phi^{a\,\mathrm{N}_{E/F}(\det(h))})\pi_1(h)v_1$$

$$\overset{\text{replace } h \text{ by } h^{-1}\det(h)}{=} -q^{-2}|U_2(E)|^{-1}\sum_{h\in H}\phi^a\Big(B(\mathrm{Id}_H, hh^\star)\Big)f(\phi^{a\,\mathrm{N}_{E/F}(\det(h))})\pi_1(h^{-1}\det(h))v_1.$$

Let

$$\kappa_a = -q^{-2}|U_2(E)|^{-1}\sum_{h\in H}\phi^a\Big(B(\mathrm{Id}_H, hh^\star)\Big)f(\phi^{a\,\mathrm{N}_{E/F}(\det(h))})\Big(\pi_1(h^{-1}\det(h))v_1\Big)(1, e_{-1}; 1)$$

and

$$\kappa_a^s = -q^{-2}|U_2(E)|^{-1}\sum_{h\in H, \mathrm{N}_{E/F}(\det(h))=s}\phi^a\Big(B(\mathrm{Id}_H, hh^\star)\Big)f(\phi^{a\,\mathrm{N}_{E/F}(\det(h))})\Big(\pi_1(h^{-1}\det(h))v_1\Big)(1, e_{-1}; 1) \text{ for any } s\in F^\times.$$

Then $\kappa_a = \sum_{s\in F^\times}\kappa_a^s$,

$$\kappa_a^1 = -q^{-2}|U_2(E)|^{-1}\sum_{h\in\mathcal{M}}\phi^a\Big(\mathrm{N}_{E/F}(\alpha)+\mathrm{N}_{E/F}(\beta)+\mathrm{N}_{E/F}(\gamma)+\mathrm{N}_{E/F}(\delta)\Big)f(\phi^a)v_1(\delta-\gamma e_{-1}, -\beta+\alpha e_{-1}; \det(h^{-1}))$$

(21)

$$= -q^{-2}|U_2(E)|^{-1}\sum_{h\in\mathcal{M}}\phi^a\Big(\mathrm{N}_{E/F}(\alpha)+\mathrm{N}_{E/F}(\beta)+\mathrm{N}_{E/F}(\gamma)+\mathrm{N}_{E/F}(\delta)\Big)f(\phi^a)\Lambda^{1-q}(\delta-\gamma e_{-1})\Lambda^q(\alpha\delta-\beta\gamma)v_1(1, \frac{-\beta+\alpha e_{-1}}{\delta-\gamma e_{-1}}; 1),$$

where $\mathcal{M} = \{h = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in H|\,\mathrm{N}_{E/F}(\alpha\delta-\beta\gamma) = 1; -\beta+\alpha e_{-1} = u_1e_{-1}(\delta-\gamma e_{-1}), \delta-\gamma e_{-1}\neq 0 \text{ for some } u_1\in U\}$;
By equations: $\{\alpha\delta-\beta\gamma = u_2; -\beta+\alpha e_{-1} = u_1e_{-1}(\delta-\gamma e_{-1})$ and $\delta-\gamma e_{-1} = y$ where $u_1, u_2\in U, y\in E^\times\}$, we change the variables $\{\alpha,\beta,\gamma,\delta\}$ by $\{u_1, u_2, y, \gamma\}$.

By $-\beta+\alpha e_{-1}=u_1e_{-1}(\delta-\gamma e_{-1})$, we get $-\beta e_{-1}^{-1}+u_1\gamma e_{-1}=u_1\delta-\alpha$. Then:

$$(-\beta e_{-1}^{-1}+u_1\gamma e_{-1})(-\beta e_{-1}^{-1}+u_1\gamma e_{-1})^q=(u_1\delta-\alpha)(u_1\delta-\alpha)^q.$$

By calculation, we have

$$\begin{aligned}&\mathrm{N}_{E/F}(\alpha)+\mathrm{N}_{E/F}(\beta)+\mathrm{N}_{E/F}(\gamma)+\mathrm{N}_{E/F}(\delta)\\&=\mathrm{Tr}_{E/F}\Big(u_1(\alpha^q\delta-\gamma\beta^qe_{-1}^{1-q})\Big).\end{aligned}$$

Set $A=\alpha^q\delta-\gamma\beta^qe_{-1}^{1-q}$. Now we consider

$$\begin{aligned}&u_1u_2^{-1}y=u_1u_2^qy\\&=u_1\Big(\alpha^q\delta^{q+1}-\beta^q\gamma^q\delta-\alpha^q\delta^q\gamma e_{-1}+\beta^q\gamma^{q+1}e_{-1}\Big),\end{aligned}$$

also

$$\begin{aligned}u_1Ay^q&=u_1[\Big(\alpha^q\delta-\gamma\beta^qe_{-1}^{1-q}\Big)\Big(\delta^q-\gamma^qe_{-1}^q\Big)]\\&=u_1[\alpha^q\delta^{q+1}-\alpha^q\delta\gamma^qe_{-1}^q-\delta^q\beta^q\gamma e_{-1}^{1-q}+\beta^q\gamma^{q+1}e_{-1}]\\&=u_1[u_2^{-1}y+\beta^q\gamma^q\delta+\alpha^q\delta^q\gamma e_{-1}-\alpha^q\delta\gamma^qe_{-1}^q-\delta^q\beta^q\gamma e_{-1}^{1-q}]\\&=u_1u_2^{-1}y+u_1e_{-1}^{-q}(-\beta+\alpha e_{-1})^q(\delta^q\gamma e_{-1}-\delta\gamma^qe_{-1}^q)\\&=u_1u_2^{-1}y+y^q(\delta^q\gamma e_{-1}-\delta\gamma^qe_{-1}^q).\end{aligned}$$

So

$$u_1A=u_1u_2^{-1}y^{1-q}+\Big(\delta^q\gamma e_{-1}-\delta\gamma^qe_{-1}^q\Big).$$

In this way, we obtain:

$$\mathrm{Tr}_{E/F}(u_1A)=\mathrm{Tr}_{E/F}(u_1u_2^{-1}y^{1-q}).$$

Hence

$$\begin{aligned}(21)&=-q^{-2}|U_2(E)|^{-1}\sum_{u_1,u_2\in U,y\in E^\times,\gamma\in E}\phi^a(\,\mathrm{Tr}_{E/F}(u_1u_2^{-1}y^{1-q}))\Lambda(u_1u_2^{-1}y^{1-q})f(\phi^a)\\&\overset{u_1=x_1^{1-q},u_2=x_2^{q-1}}{=}-q^{-2}|U_2(E)|^{-1}\frac{1}{(q-1)^2}\sum_{x_1,x_2,y\in E^\times,\gamma\in E}\phi^a\Big(\,\mathrm{Tr}_{E/F}\Big((x_1x_2y)^{1-q}\Big)\Big)\Lambda\Big((x_1x_2y)^{1-q}\Big)f(\phi^a)\\&=-\frac{1}{(q-1)q}\sum_{y\in E^\times}\phi^a\Big(\,\mathrm{Tr}_{E/F}\,(y^{1-q})\Big)\Lambda\Big(y^{1-q}\Big)f(\phi^a)=-q^{-1}\sum_{y\in E^\times,\mathrm{N}_{E/F}(y)=1}\phi^a(\mathrm{Tr}_{E/F}(y))\Lambda(y)f(\phi^a).\end{aligned}$$

Similarly we obtain

$$\kappa_a^s=-q^{-1}\sum_{y\in E^\times,\mathrm{N}_{E/F}(y)=s}\phi^a(\mathrm{Tr}_{E/F}(y))\Lambda(y)f(\phi^{a\mathrm{N}_{E/F}(y)}).$$

Finally

$$\kappa_a=-q^{-1}\sum_{y\in E^\times}\phi^a(\mathrm{Tr}_{E/F}(y))\Lambda(y)f(\phi^{aN_{E/F}(y)}).$$

Since $V_1^{U_2(E)}$ is one dimension, it implies that:

$$(\pi_0(\omega)j(f))(\xi_a)=j(\pi_\Lambda(\omega)f)(\xi_a),$$

which means $\pi_0(\omega)j(f)=j(\pi_\Lambda(\omega)f)$.
By the above (1), (2), we prove $\pi_0\simeq\pi_\Lambda$.

3.7. By the above discussion I—IV about the representation $(\pi_0,W[\pi_1])$, finally we achieve the main theorem in this section:

Theorem 3.9. *For the representation $(\pi,G\times H,W)$, we have the following decomposition:*

$$\pi=\bigoplus_{\sigma\in\mathrm{Irr}(G)}\sigma\otimes\mathrm{Bc}_{E/F}(\sigma)\oplus\bigoplus_{\psi\in\mathrm{Irr}(F^\times),\Psi\in\mathrm{Irr}(E^\times),\Psi=\psi\circ\mathrm{N}_{E/F}}\psi\mathrm{St}_G\otimes\Psi\cdot 1_H.$$

4. The decomposition of the Weil representation of $\mathrm{GL}_2(K)$

4.1. In this section, we use the following notations: $G = \mathrm{GL}_2(K)$, $B = \{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in G\}$, $N = \{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in G\}$, $T = \{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in G\}$, $Z = \{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in G\}$; $\mathrm{Gal}(K/F) = \langle \sigma \rangle$; $u(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ for $b \in K$, $h(a,d) = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ for $a, d \in K^\times$, $\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in G$.

4.2. We recall the technique of Weil's Galois descent to construct a morphism from G to $\mathrm{GSp}_8(F)$.

Let V_0 be a vector space over F of dimension 2, endowed with a non-degenerate symplectic form $\langle, \rangle_{V_0}$. Let $\{e_1, e_2\}$ be a symplectic base of V_0. Namely $V = V_0 \otimes_F K$ is a symplectic K-vector space, endowed with the symplectic form $\langle, \rangle_V$ induced from V_0. Let us define a $\mathrm{Gal}(K/F)$-action on V by

$$\mathrm{Gal}(K/F) \times K \otimes_F V_0 \longrightarrow K \otimes_F V_0; (\sigma, \sum_i k_i \otimes e_i) \longmapsto \sum_i k_i^\sigma \otimes e_i.$$

Now let $W = V \otimes_K V \otimes_K V$, by following the symplectic structure of V, we associate W a symplectic form $\langle, \rangle_V \otimes \langle, \rangle_V \otimes \langle, \rangle_V$. On W, we will consider the twisted Galois action defined by

$$\mathrm{Gal}(K/F) \times W \longrightarrow W; (\sigma, w = \sum_i u_i \otimes v_i \otimes w_i) \longmapsto {}^\sigma w = \sum_i w_i^\sigma \otimes u_i^\sigma \otimes v_i^\sigma.$$

We will let W_0 denote the set $\{w \in W | {}^\sigma w = w\}$. By calculation, each $w_0 \in W_0$ may be expressed in the form

$$w_0 = x e_1 \otimes e_1 \otimes e_1 + \alpha e_1 \otimes e_1 \otimes e_2 + \alpha^\sigma e_2 \otimes e_1 \otimes e_1 + \alpha^{\sigma^2} e_1 \otimes e_2 \otimes e_1$$

$$+ \beta^{\sigma^2} e_2 \otimes e_1 \otimes e_2 + \beta^\sigma e_1 \otimes e_2 \otimes e_2 + \beta e_2 \otimes e_2 \otimes e_1 + y e_2 \otimes e_2 \otimes e_2 \text{ for } x, y \in F; \alpha, \beta \in K.$$

Every element w_0 of this form is well-defined by its corresponding coefficients. By abuse of notation, we write $w_0 = \begin{pmatrix} x & \alpha \\ \beta & y \end{pmatrix}$ instead of the whole term. One can check that the symplectic form over the F-vector space W_0 is a F-symplectic form, denoted by $\langle, \rangle_{W_0}$. More precisely,

$$\langle w_0, w_0' \rangle_{W_0} = xy' - x'y - \mathrm{Tr}_{K/F}(\alpha\beta') + \mathrm{Tr}_{K/F}(\alpha'\beta) \text{ for } w_0 = \begin{pmatrix} x & \alpha \\ \beta & y \end{pmatrix}, w_0' = \begin{pmatrix} x' & \alpha' \\ \beta' & y' \end{pmatrix}.$$

Let $\mathrm{GSp}(W)$ denote the symplectic similitude group associated to W. By the definition of W, actually, there exists a morphism of groups $\Big(\mathrm{GL}(V) \times \mathrm{GL}(V) \times \mathrm{GL}(V)\Big) \rtimes S_3 \longrightarrow \mathrm{GSp}(W)$. The group S_3 acts on W by permutating its three variables. Now, we define a twisted Galois action of $\mathrm{Gal}(K/F)$ on $\mathrm{GL}(V) \times \mathrm{GL}(V) \times \mathrm{GL}(V)$ by

$$\mathrm{Gal}(K/F) \times \Big(\mathrm{GL}(V) \times \mathrm{GL}(V) \times \mathrm{GL}(V)\Big) \longrightarrow \mathrm{GL}(V) \times \mathrm{GL}(V) \times \mathrm{GL}(V); h = (g_1, g_2, g_3) \longmapsto {}^\sigma h := (g_3^\sigma, g_1^\sigma, g_2^\sigma).$$

Write $\overline{\mathrm{GL}(V)} = \{h \in \mathrm{GL}(V) \times \mathrm{GL}(V) \times \mathrm{GL}(V) | {}^\sigma h = h\}$. So there exists an isomorphism of groups $\mathrm{GL}(V) \longrightarrow \overline{\mathrm{GL}(V)}; g \longmapsto (g, g^\sigma, g^{\sigma^2})$. If given $h \in \mathrm{GL}(V) \times \mathrm{GL}(V) \times \mathrm{GL}(V)$, $w \in W = V \otimes_K V \otimes_K V$, one can verify ${}^\sigma h \cdot {}^\sigma w = {}^\sigma(h \cdot w)$. So it induces a morphism from $\mathrm{GL}(V) \simeq \overline{\mathrm{GL}(V)}$ to $\mathrm{GSp}(W_0)$. By the fixed basis $\{e_1, e_2\}$, in fact, we obtain a morphism $i : G \longrightarrow \mathrm{GSp}(W_0)$.

4.3. We interpret the above construction of the morphism $G \xrightarrow{i} \mathrm{GSp}(W_0)$ in terms of the language of algebraic groups.

Let $\mathbf{V}$ be the K-algebraic vector space associated to V. That is to say $\mathbf{V} : \mathbf{Alg}_K \longrightarrow \mathbf{Vect}_K; R \longmapsto V \otimes_K R$, a functor from the category of unital commutative associate K-algebras to the category of K-vector spaces. Namely $V \otimes_K R$ inherits the R-symplectic structure from V. We could define a $\mathrm{Gal}(K/F)$-action on $\mathbf{V}$ in the following way:

$$\mathrm{Gal}(K/F) \times V \otimes_K R \longrightarrow V \otimes_K R; (\sigma, \sum_{i=1}^n v_i \otimes r_i) \longmapsto \sum_{i=1}^n v_i^\sigma \otimes r_i^\sigma.$$

Now let $\mathbf{W}$ be the K-algebraic vector space associated to W, and $\mathbf{W}_0$ the F-algebraic vector space associated to W_0. We give a twisted $\mathrm{Gal}(K/F)$-action on $\mathbf{W}$ in the following way:

$$\mathrm{Gal}(K/F) \times V \otimes_K V \otimes_K V \otimes_K R \longrightarrow V \otimes_K V \otimes_K V \otimes_K R;$$
$$(\sigma, \sum_{i=1}^n u_i \otimes v_i \otimes w_i \otimes r_i) \longmapsto \sum_{i=1}^n w_i^\sigma \otimes u_i^\sigma \otimes v_i^\sigma \otimes r_i^\sigma.$$

So $\mathbf{W}_0$ is the $\mathrm{Gal}(K/F)$-invariant algebraic scheme of $\mathbf{W}$ in the following sense:

(1) $\mathbf{W} \simeq \mathbf{W}_0 \times_F K$.

(2) $\mathbf{W}(R)^{\mathrm{Gal}(K/F)} \simeq \mathbf{W}_0(R^{\mathrm{Gal}(K/F)})$ for any $R \in \mathbf{Alg}_K$.

On the other hand, we also define a twisted Galois action of $\mathrm{Gal}(K/F)$ on $\mathbf{GL}_{2/K} \times \mathbf{GL}_{2/K} \times \mathbf{GL}_{2/K}$ in the following way:

$$\mathrm{Gal}(K/F) \times \Big(\mathrm{GL}_2(R) \times \mathrm{GL}_2(R) \times \mathrm{GL}_2(R)\Big) \longrightarrow \mathrm{GL}_2(R) \times \mathrm{GL}_2(R) \times \mathrm{GL}_2(R);$$
$$(\sigma, (g_1, g_2, g_3)) \longmapsto (g_3^\sigma, g_2^\sigma, g_1^\sigma).$$

We denote by $\mathbf{H}^{\mathrm{Gal}(K/F)}$ the $\mathrm{Gal}(K/F)$-invariant algebraic scheme of $\mathbf{H} = \mathbf{GL}_{2/K} \times \mathbf{GL}_{2/K} \times \mathbf{GL}_{2/K}$. Indeed, by definitions, $\mathbf{H}^{\mathrm{Gal}(K/F)} \simeq \mathrm{Res}_{K/F}(\mathbf{GL}_{2/K})$. There exists an action of $\mathbf{H}^{\mathrm{Gal}(K/F)}$ on $\mathbf{W}_0$, and it preserves the symplectic form up to the similitudes. Thus we could obtain a morphism of algebraic group schemes:

$$\mathbf{i} : \mathrm{Res}_{K/F}(\mathbf{GL}_{2/K}) \longrightarrow \mathbf{GSp}_{W_0}.$$

4.4. Let $X_0 = \{w_0 = \begin{pmatrix} x & \alpha \\ 0 & 0 \end{pmatrix} | w_0 \in W_0\}$, $Y_0 = \{w_0 = \begin{pmatrix} 0 & 0 \\ \beta & y \end{pmatrix} | w_0 \in W_0\}$. Then X_0, Y_0 are two vector spaces over F and $W_0 = X_0 \oplus Y_0$ is a complete polarization of W_0. Via the morphism $i : G \longrightarrow \mathrm{GSp}(W_0)$, it gives rise to a G-action on W_0 by the following formulas:

for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$, $w_0 = \begin{pmatrix} x & \alpha \\ \beta & y \end{pmatrix}$, write $g \cdot w_0 = \begin{pmatrix} x' & \alpha' \\ \beta' & y' \end{pmatrix}$, then

$x' = \mathrm{N}_{K/F}(a)x + \mathrm{Tr}_{K/F}(aa^\sigma b^{\sigma^2}\alpha) + \mathrm{Tr}_{K/F}(bb^\sigma a^{\sigma^2}\beta) + \mathrm{N}_{K/F}(b)y$;
$\alpha' = aa^\sigma c^{\sigma^2} x + (aa^\sigma d^{\sigma^2}\alpha + ba^\sigma c^{\sigma^2}\alpha^\sigma + ab^\sigma c^{\sigma^2}\alpha^{\sigma^2}) + (bb^\sigma c^{\sigma^2}\beta + ab^\sigma d^{\sigma^2}\beta^\sigma + ba^\sigma d^{\sigma^2}\beta^{\sigma^2}) + bb^\sigma d^{\sigma^2} y$;
$\beta' = dd^\sigma b^{\sigma^2} y + (dd^\sigma a^{\sigma^2}\beta + cd^\sigma b^{\sigma^2}\beta^\sigma + dc^\sigma b^{\sigma^2}\beta^{\sigma^2}) + (cc^\sigma b^{\sigma^2}\alpha + dc^\sigma a^{\sigma^2}\alpha^\sigma + cd^\sigma a^{\sigma^2}\alpha^{\sigma^2}) + cc^\sigma a^{\sigma^2} x$;
$y' = \mathrm{N}_{K/F}(d)y + \mathrm{Tr}_{K/F}(dd^\sigma c^{\sigma^2}\beta) + \mathrm{Tr}_{K/F}(cc^\sigma d^{\sigma^2}\alpha) + \mathrm{N}_{K/F}(c)x$.

We write each element $g \in \mathrm{GSp}(W_0)$ in the form of $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $a \in \mathrm{End}_F(X_0), b \in \mathrm{Hom}_F(Y_0, X_0), c \in \mathrm{Hom}_F(X_0, Y_0), d \in \mathrm{End}_F(Y_0)$.

Corollary 4.1. *Through the map $i : G \longrightarrow \mathrm{GSp}(W_0)$, the actions of $u(b), h(a, d), \omega$ on W_0 are described as follows:*

(1) $i(u(b)) = \begin{pmatrix} m & n \\ 0 & m^\vee \end{pmatrix}$ where $b \in K$, $m \in \mathrm{End}_F(X_0)$, $n \in \mathrm{Hom}_F(Y_0, X_0)$, $m^\vee \in \mathrm{End}_F(Y_0)$ is the contragredient of m for the symplectic form $\langle, \rangle$ and $m\begin{pmatrix} x & \alpha \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x + \mathrm{Tr}_{K/F}(b^{\sigma^2}\alpha) & \alpha \\ 0 & 0 \end{pmatrix}$; $n\begin{pmatrix} 0 & 0 \\ \beta & y \end{pmatrix} = \begin{pmatrix} \mathrm{Tr}_{K/F}(bb^\sigma\beta) + \mathrm{N}_{K/F}(b)y & b^\sigma\beta^\sigma + b\beta^{\sigma^2} + bb^\sigma y \\ 0 & 0 \end{pmatrix}$; $m^\vee\begin{pmatrix} 0 & 0 \\ \beta & y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \beta + b^{\sigma^2}y & y \end{pmatrix}$;

(2) $i(h(a, d)) = \begin{pmatrix} m & 0 \\ 0 & n \end{pmatrix}$ where $m\begin{pmatrix} x & \alpha \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \mathrm{N}_{K/F}(a)x & aa^\sigma d^{\sigma^2}\alpha \\ 0 & 0 \end{pmatrix}$, $n\begin{pmatrix} 0 & 0 \\ \beta & y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ dd^\sigma a^{\sigma^2}\beta & \mathrm{N}_{K/F}(d)y \end{pmatrix}$;

(3) $i(\omega) = \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}$ where $u\begin{pmatrix} 0 & 0 \\ \beta & y \end{pmatrix} = \begin{pmatrix} y & -\beta \\ 0 & 0 \end{pmatrix}$; $v\begin{pmatrix} x & \alpha \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \alpha & -x \end{pmatrix}$.

Let (ρ, V) be the Weil representation of the symplectic similitude group $\mathrm{GSp}(W_0)$. Via the map i, it gives rise to a representation (π, V) of G. The representation π can be realized in the vector space $V = \mathbb{C}[Y_0 \times X_F]$ of complex functions on $Y_0 \times X_F$.

Proposition 4.2. *For the representation $(\pi, G, \mathbb{C}[Y_0 \times X_F])$, the action is determined by the following formulas:*

(1) $[\pi(u(b))F]\Big(\begin{pmatrix} 0 & 0 \\ \beta & y \end{pmatrix}, \psi\Big) = \psi\Big(\mathrm{Tr}_{K/F}(bb^\sigma\beta y) - \mathrm{N}_{K/F}(b)y^2 - \mathrm{Tr}_{K/F}(b\beta\beta^{\sigma^2})\Big)F\Big(\begin{pmatrix} 0 & 0 \\ \beta - b^{\sigma^2}y & y \end{pmatrix}, \psi\Big)$;

(2) $[\pi(h(a,d))F]\Big(\begin{pmatrix} 0 & 0 \\ \beta & y \end{pmatrix}, \psi\Big) = \chi_q^+(\mathrm{N}_{K/F}(ad))F\Big(\begin{pmatrix} 0 & 0 \\ \frac{\mathrm{N}_{K/F}(ad)}{dd^{\sigma}a^{\sigma^2}}\beta & \mathrm{N}_{K/F}(a)y \end{pmatrix}, \psi^{\mathrm{N}_{K/F}(ad)^{-1}}\Big)$;

(3) $[\pi(\omega)F]\Big(\begin{pmatrix} 0 & 0 \\ \beta & y \end{pmatrix}, \psi\Big) = q^{-2}\sum_{\beta' \in K, y' \in F} F\Big(\begin{pmatrix} 0 & 0 \\ \beta' & y' \end{pmatrix}, \psi\Big)\psi(yy' + \mathrm{Tr}_{K/F}(\beta\beta'))$.

Proof. See Appendix 1. □

4.5. The whole goal of this section is to determine the isotypic components of the representation π. We first consider the principal series representations.

Let $\alpha, \beta \in \mathrm{Irr}(K^\times)$. To determine the principal series components of π, it involves to calculate the dimension of the vector space $\mathrm{Hom}_G(V, \mathrm{Ind}_B^G(\alpha \otimes \beta))$. Applying the Frobenius Reciprocity Theorem, we see $\mathrm{Hom}_G(V, \mathrm{Ind}_B^G(\alpha \otimes \beta)) \simeq \mathrm{Hom}_T(V_N, \alpha \otimes \beta) \simeq \mathrm{Hom}_T(V^N, \alpha \otimes \beta)$. Therefore we should first describe the vector space V^N, then consider the T-action on it.

Once we regard the action of N on the vector space V, as described in Proposition 4.2 (1), we should consider the action:

$$N \times (Y_0 \times X_F) \longrightarrow Y_0 \times X_F; \Big(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \times \Big(\begin{pmatrix} 0 & 0 \\ \beta & y \end{pmatrix}, \psi\Big)\Big) \longmapsto \Big(\begin{pmatrix} 0 & 0 \\ \beta - b^{\sigma^2}y & y \end{pmatrix}, \psi\Big).$$

The orbits of this action are following:

(i) $\mathrm{Orbit}\{\xi_{(\beta,0;\psi)}\}$ where $\xi_{(\beta,0;\psi)} = \Big(\begin{pmatrix} 0 & 0 \\ \beta & 0 \end{pmatrix}, \psi\Big)$ for any $\beta \in K, \psi \in X_F$;

(ii) $\mathrm{Orbit}\{\eta_{(0,y;\psi)}\}$ where $\eta_{(0,y;\psi)} = \Big(\begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix}, \psi\Big)$ for any $y \in F^\times, \psi \in X_F$.

The stabilizer of the chosen element in each orbit is following:

(i) $\mathrm{Stab}_N(\xi_{(\beta,0;\psi)}) = N$;
(ii) $\mathrm{Stab}_N(\eta_{(0,y;\psi)}) = 1_N$.

Let F be a function in the vector space V. Then it belongs to V^N if and only if it satisfies the equality:

$$\psi(\mathrm{Tr}_{K/F}(bb^{\sigma}\beta y) - \mathrm{N}_{K/F}(b)y^2 - \mathrm{Tr}_{K/F}(b\beta\beta^{\sigma^2}))F\Big(\begin{pmatrix} 0 & 0 \\ \beta - b^{\sigma^2}y & y \end{pmatrix}, \psi\Big) = F\Big(\begin{pmatrix} 0 & 0 \\ \beta & y \end{pmatrix}, \psi\Big). \tag{22}$$

for any $b \in K$.

Proposition 4.3. *(1) The vector space V^N is generated by the following functions:*

(i) $F_{(0,0;\psi)}$ *$\mathrm{supp}(F_{(0,0;\psi)}) = $ Orbit $\{\xi_{(0,0;\psi)}\}$ and $F_{(0,0,\psi)}(\xi_{(0,0;\psi)}) = 1$, it satisfies equality (22) for any $\psi \in X_F$;*
(ii) $G_{(0,y;\psi)}$ *$\mathrm{supp}(G_{(0,y;\psi)}) = $ Orbit $\{\eta_{(0,y;\psi)}\}$ and $G_{(0,y;\psi)}(\eta_{(0,y;\psi)}) = 1$, it satisfies equality (22) for any $y \in F^\times, \psi \in X_F$.*

(2) Let $t = h(a,d) \in T$. Then the action of t on the vector space V^N is given as follows:

(i) $\pi(t)F_{(0,0;\psi)} = \chi_q^+(\mathrm{N}_{K/F}(ad))F_{(0,0;\psi^{\mathrm{N}_{K/F}(ad)})}$;
(ii) $\pi(t)G_{(0,y;\psi)} = \chi_q^+(\mathrm{N}_{K/F}(ad))G_{(0,\frac{1}{\mathrm{N}_{K/F}(a)}y;\psi^{\mathrm{N}_{K/F}(ad)})}$.

Proof. (1) Every element F in V^N, that satisfies the (22), is completely determined by its values over the points $\{\xi_{(\beta,0;\psi)}, \eta_{(0,y;\psi)}\}$. Let δ be one point among them. Then $F(\delta)$ can be nonzero if and only if the coefficient on the right hand in equality (22) is trivial over the stabilizer of δ, After checking every such point, we obtain the result.
(2) It is straightforward. □

Let Φ be an element in $\mathrm{Hom}_T(V^N, \alpha \boxtimes \beta)$, it is determined by the following two equations:

$$\chi_q^+(\mathrm{N}_{K/F}(ad))\Phi(F_{(0,0;\psi^{\mathrm{N}_{K/F}(ad)})}) = \alpha(a)\beta(b)\Phi(F_{(0,0;\psi)}), \tag{23}$$

$$\chi_q^+(\mathrm{N}_{K/F}(ad))\Phi(G_{(0,\frac{1}{\mathrm{N}_{E/F}(a)}y;\psi^{\mathrm{N}_{K/F}(ad)})}) = \alpha(a)\beta(b)\Phi(G_{(0,y;\psi)}). \tag{24}$$

for $a, d \in K^\times$.

We define a T-action on the vector space V^N: for any $t = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$,

(i) $t \cdot F_{(0,0;\psi)} := F_{(0,0;\psi^{\mathrm{N}_{K/F}(ad)})}$;
(ii) $t \cdot G_{(0,y;\psi)} = G_{(0,\frac{1}{\mathrm{N}_{E/F}(a)}y;\psi^{\mathrm{N}_{E/F}(ad)})}$.

For such action, there are two kinds of orbits:

(i) Orbit $\{F_{(0,0;\phi)}\}$;
(ii) Orbit $\{G_{(0,1;\phi)}\}$ for one fixing $\phi \in X_F$.

The stabilizer of the representative element in each orbit is following:

(i) $\mathrm{Stab}_T(F_{(0,0;\phi)}) = \{h(a,d) \in T | \mathrm{N}_{K/F}(ad) = 1\}$;
(ii) $\mathrm{Stab}_T(G_{(0,1;\phi)}) = \{h(a,d) \in T | \mathrm{N}_{K/F}(a) = \mathrm{N}_{K/F}(d) = 1\}$.

Now we get the following statement relative to the principal series components of the representation π:

Proposition 4.4. *Let* $\alpha, \beta \in \mathrm{Irr}(K^\times)$.
(1) If $\alpha = \chi_1 \circ \mathrm{N}_{K/F}, \beta = \chi_2 \circ \mathrm{N}_{K/F}$ *for some characters* $\chi_1 \neq \chi_2 \in \mathrm{Irr}(F^\times)$*, then* $\dim_{\mathbb{C}} \mathrm{Hom}_G(V, \mathrm{Ind}_B^G(\alpha \otimes \beta)) = 1$.
(2) If $\alpha = \beta = \chi \circ \mathrm{N}_{K/F}$ *for a character* $\chi \in \mathrm{Irr}(F^\times)$*, then* $\dim_{\mathbb{C}} \mathrm{Hom}_G(V, \mathrm{Ind}_B^G(\alpha \cdot 1_B)) = 2$.
(3) For the other kind of $\alpha, \beta \in \mathrm{Irr}(K^\times)$*,* $\mathrm{Hom}_G(V, \mathrm{Ind}_B^G(\alpha \otimes \beta)) = 0$.

Proof. By Frobenius Reciprocity Theorem, we see $\mathrm{Hom}_{\mathbb{C}}(V, \mathrm{Ind}_B^G(\alpha \otimes \beta)) \simeq \mathrm{Hom}_T(V^N, \alpha \otimes \beta)$. Let $\Phi \in \mathrm{Hom}_T(V^N, \alpha \otimes \beta)$. The function Φ is completely determined by its values on the points $F_{(0,0;\phi)}$ and $G_{(0,1;\phi)}$. The values $\Phi(F_{(0,0;\phi)})$ can be any complex number iff $\alpha \otimes \beta(t) = 1$ for $t = h(a,d) \in \mathrm{Stab}_T(F_{(0,0;\phi)})$ which is equivalent to $\alpha = \beta = \chi \circ \mathrm{N}_{K/F}$ for some character $\chi \in \mathrm{Irr}(F^\times)$. Similarly the values $\Phi(G_{(0,1;\phi)})$ can be any complex number iff $\alpha = \chi_1 \circ \mathrm{N}_{K/F}, \beta = \chi_2 \circ \mathrm{N}_{K/F}$ for characters $\chi_1, \chi_2 \in \mathrm{Irr}(F^\times)$, thus we obtain the result. □

4.6. Now it reduces to check whether the representation $\chi \circ \mathrm{N}_{E/F} \cdot 1_G$ is a sub-representation of π. Let $(\alpha \cdot \pi, V_\alpha)$ be the twisted representation of π by the character $\alpha = \chi \circ \mathrm{N}_{K/F} \in \mathrm{Irr}(G)$. Since $\mathrm{Hom}_G(\pi, \alpha^{-1} \cdot 1_G) \simeq V_\alpha^G$. It is enough to determine the dimension of V_α^G for the representation $(\alpha \cdot \pi, G, V_\alpha)$. First we notice that $V_\alpha^N \simeq V^N$ which is generated by functions $\{F_{(0,0;\psi)}, G_{(0,y;\psi)}\}$ and the action of T on V_α^N is given by the following formulas:
(1) $(\alpha \cdot \pi)(h(a,d))F_{(0,0;\psi)} = \chi \cdot \chi_q^+(\mathrm{N}_{K/F}(ad))F_{(0,0;\psi^{\mathrm{N}_{E/F}(ad)})}$;
(2)$(\alpha \cdot \pi)(h(a,d))G_{(0,y;\psi)} = \chi \cdot \chi_q^+(\mathrm{N}_{K/F}(ad))G_{(0,\frac{1}{\mathrm{N}_{K/F}(a)}y;\psi^{\mathrm{N}_{K/F}(ad)})}$.

Proposition 4.5. *The vector space* V_α^B *is generated by two non-zero functions* $A = \sum_{t \in T} \alpha \cdot \pi(t) F_{(0,0;\phi)}$ *and* $B = \sum_{t \in T} \alpha \cdot \pi(t) G_{(0,1;\phi)}$ *for the fixed* $\phi \in X_F$.

Proof. It is straightforward. □

The finally key step is to consider the action of ω on the vector space V_α^B. Observe that $\alpha \cdot \pi(\omega)A, \alpha \cdot \pi(\omega)B$ both belong to V_α^T. We treat the vector space V_α^N similarly as V_α^T. Consider the T-action on the set $Y_0 \times X_F$:

$$T \times (Y_0 \times X_F) \longrightarrow Y_0 \times X_F; \left(\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \times \left(\begin{pmatrix} 0 & 0 \\ \beta & y \end{pmatrix}, \psi \right) \right) \longmapsto \left(\begin{pmatrix} 0 & 0 \\ \frac{\mathrm{N}_{K/F}(ad)}{dd^\sigma a^{\sigma^2}} & \mathrm{N}_{K/F}(a)y \end{pmatrix}, \psi^{\mathrm{N}_{K/F}(ad)^{-1}} \right).$$

The orbits of this action are following:

(i) Orbit $\{x_{0,0}\}$ for $x_{0,0} = \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \phi \right)$,

(ii) Orbit $\{x_{1,0}\}$ for $x_{1,0} = \left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \phi \right)$,

(iii) Orbit $\{x_{0,1}\}$ for $x_{0,1} = \left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \phi \right)$,

(iv) Orbit $\{y_k\}$ for $y_k = \left(\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \phi^k \right)$,

for the fixed character $\phi \in X_F$, and any $k \in F^\times$.
Therefore, for the functions $A, B, \alpha \cdot \pi(\omega)A, \alpha \cdot \pi(\omega)B$, we only need to determine their values on the above four kinds of points: (1) $x_{0,0}$; (2) $x_{1,0}$; (3) $x_{0,1}$; (4) y_k. By the calculations in Appendix 2, we obtain the following table for the values of the functions $A, B, \alpha \cdot \pi(\omega)A, \alpha \cdot \pi(\omega)B$ over the above points:

	x_{00}	x_{01}	x_{10}	y_k
A	$(q-1)(q^2+q+1)^2$	0	0	0
B	0	$(q^2+q+1)^2$	0	$\chi\chi_q^+(k)\phi(-k)(q^2+q+1)^2$
$\alpha\cdot\pi(\omega)A$	$q^{-2}(q-1)(q^2+q+1)^2$	$q^{-2}(q-1)(q^2+q+1)^2$	$q^{-2}(q-1)(q^2+q+1)^2$	$\chi\chi_q^+(k)q^{-2}(q-1)(q^2+q+1)^2$
$\alpha\cdot\pi(\omega)B$	$-q^{-1}(q-1)(q+1)(q^2+q+1)^2$	$q^{-1}(q+1)(q^2+q+1)^2$	$q^{-1}(q^2+q+1)^2$	

Corollary 4.6. *The element $qA-(q-1)B\in V_\alpha^B$ is $\pi(\omega)$-invariant.*

Proof. Let us consider $C=\sum_{g\in G}\alpha\cdot\pi(g)F_{(0,0;\phi)}$. Then $C(x_{0,0})=\sum_{n\in N,b\in B}\alpha\cdot\pi(n)\alpha\cdot\pi(\omega)\alpha\cdot\pi(b)F_{(0,0;\phi)}(x_{0,0})+\sum_{b\in B}\alpha\cdot\pi(b)F_{(0,0;\phi)}(x_{0,0})=q^3[\sum_{n\in N}\alpha\cdot\pi(n)\alpha\cdot\pi(\omega)A+A](x_{0,0})=q^3[q^3\alpha\cdot\pi(\omega)A(x_{0,0})+A(x_{0,0})]=q^3(q+1)(q-1)(q^2+q+1)^2\neq 0$. As $\pi(\omega)A\neq A$, this means that $\dim V_\alpha^G=1$. So there exists two constants $a,b\in\mathbb{C}^\times$ such that $aA+bB$ is $\pi(\omega)$-invariant. By the above diagram, we can set $a=q, b=-(q-1)$. □

Corollary 4.7. *For any character $\chi\in\mathrm{Irr}(F^\times)$ and $\alpha^{-1}=\chi^{-1}\circ\mathrm{N}_{K/F}$:*
(1) $\dim_{\mathbb{C}}\mathrm{Hom}_{\mathbb{C}}(V,\alpha^{-1}\cdot 1_G)=1$;
(2) $\dim_{\mathbb{C}}\mathrm{Hom}_{\mathbb{C}}(V,\alpha^{-1}\cdot\mathrm{St}_G)=1$.

Proof. (1) $\mathrm{Hom}_{\mathbb{C}}(V,\alpha^{-1}\cdot\mathrm{St}_G)\simeq V_\alpha^G$ and $\pi(\omega)A\neq A$, $\pi(\omega)(qA-(q-1)B)=qA-(q-1)B$, it implies $\dim_{\mathbb{C}}V_\alpha^G=1$.
(2) It follows from above (1) and Proposition 4.4. □

Proposition 4.8. *The non cuspidal part of the Weil representation π is presented as follows:*

$$\pi_{non-cusp}=\bigoplus_{\sigma\in\mathrm{Irr}_{non-cusp}(\mathrm{GL}_2(F))}\mathrm{Bc}_{K/F}(\sigma),$$

where $\pi_{non-cusp}$ is the non cuspidal part of the representation π and $\mathrm{Bc}_{K/F}$ is the map of base change from $\mathrm{Irr}(\mathrm{GL}_2(F))$ *to* $\mathrm{Irr}(\mathrm{GL}_2(K))$.

Proof. It follows from Proposition 1.4(2), Proposition 4.4 and Corollary 4.7. □

4.7. We continue the above discussion and determine the cuspidal part of π.

Now let K_1 (resp. F_1) be a quadratic field extension of K (resp. F). Assume $K_1\supset F_1$. Let ρ(resp. ρ_1) denote the Weil representation of $\mathbf{GSp}_{W_0}(F)$(resp. $\mathbf{GSp}_{W_0}(F_1)$). Denote by $\pi=\rho|_{\mathrm{GL}_2(K)}$ and $\pi_1=\rho_1|_{\mathrm{GL}_2(K_1)}$.

By Theorem 1.7 in §1.5, there exists a unique representation $\widetilde{\rho_1}$ of the group $\mathrm{Gal}(F_1/F)\ltimes\mathbf{GSp}_{W_0}(F_1)$ such that 0-res$(\widetilde{\rho_1})=\rho_1$, and 1-res$(\widetilde{\rho_1})=\rho$. By the result in §4.2, there exists a morphism from $Res_{K/F}(\mathbf{GL}_2)$ to $\mathbf{GSp}_{W_0}$, which induces a map $\widetilde{p}:\mathrm{Gal}(K_1/K)\ltimes\mathbf{GL}_2(K_1)\simeq\mathrm{Gal}(F_1/F)\ltimes Res_{K/F}(\mathbf{GL}_2)(F_1)\longrightarrow\mathrm{Gal}(F_1/F)\ltimes\mathbf{GSp}_{W_0}(F_1)$. Via the map $\widetilde{p}_1$, let $\widetilde{\pi_1}=\widetilde{\rho_1}|_{\mathrm{Gal}(K_1/K)\ltimes\mathbf{GL}_2(K)}$. By the definition of i-restriction, one sees 0-res$(\widetilde{\pi_1})=\pi_1$ and 1-res$(\widetilde{\pi_1})=\pi$.

For a cuspidal representation Π_Λ of $\mathrm{GL}_2(K)$, we know $\mathrm{Bc}_{K_1/K}(\Pi_\Lambda)=\Pi_{\Lambda,\Lambda^{q^3}}$ by Proposition 1.4. Let $\widetilde{\Pi_{\Lambda,\Lambda^{q^3}}}$ denote the unique representation of the group $\mathrm{Gal}(K_1/K)\ltimes\mathbf{GL}_2(K_1)$ such that 0-res$(\widetilde{\Pi_{\Lambda,\Lambda^{q^3}}})$ and 1-res $(\widetilde{\Pi_{\Lambda,\Lambda^{q^3}}})=\Pi_\Lambda$. By Proposition 4.4, $\langle\pi_1,\Pi_{\Lambda,\Lambda^{q^3}}\rangle_{\mathrm{GL}_2(K_1)}=1$ for $\Lambda=\lambda\circ\mathrm{N}_{K_1/F_1}$ where λ is a regular character of $F_1^\times$. And by Lemma 1.5(i),

$$\begin{aligned}&\langle\widetilde{\pi_1},\widetilde{\Pi_{\Lambda,\Lambda^{q^3}}}\rangle_{\mathrm{Gal}(K_1/K)\ltimes\mathbf{GL}_2(K_1)}\\
&=\frac{1}{|\mathrm{Gal}(K/F)\ltimes\mathrm{GL}_2(K_1)|}\Big(\sum_{g\in\mathrm{GL}_2(K_1)}\chi_{\widetilde{\pi_1}}\big((1,g)\big)\overline{\chi_{\widetilde{\Pi_{\Lambda,\Lambda^{q^3}}}}\big((1,g)\big)}+\sum_{g\in\mathrm{GL}_2(K_1)}\chi_{\widetilde{\pi_1}}\big((\sigma,g)\big)\overline{\chi_{\widetilde{\Pi_{\Lambda,\Lambda^{q^3}}}}\big((\sigma,g)\big)}\Big)\\
&=\frac{|\mathrm{GL}_2(K_1)|}{|\mathrm{Gal}(K_1/K)\ltimes\mathrm{GL}_2(K_1)|}\langle\pi_1,\Pi_{\Lambda,\Lambda^{q^3}}\rangle_{\mathrm{GL}_2(K_1)}+\frac{|\mathrm{GL}_2(K_1)|}{|\mathrm{Gal}(K_1/K)\ltimes\mathrm{GL}_2(K_1)|}\langle\pi,\Pi_\Lambda\rangle_{\mathrm{GL}_2(K)}\\
&=\frac{1}{2}\Big(\langle\pi_1,\Pi_{\Lambda,\Lambda^{q^3}}\rangle_{\mathrm{GL}_2(K_1)}+\langle\pi,\Pi_\Lambda\rangle_{\mathrm{GL}_2(K)}\Big)=\frac{1}{2}\Big(1+\langle\pi,\Pi_\Lambda\rangle_{\mathrm{GL}_2(K)}\Big)\end{aligned}$$

for $\Lambda = \lambda \circ \mathrm{N}_{K_1/F_1}$. It follows that for such Λ, $\langle \pi, \Pi_\Lambda \rangle_{\mathrm{GL}_2(K)} \geq 1$. By comparing the dimension of the representation π with the total dimension of the non cuspidal representations of the group $\mathrm{GL}_2(K)$ in Proposition 4.8, we see $\langle \pi, \Pi_\Lambda \rangle_{\mathrm{GL}_2(K)} = 1$. It will also turn out that there are no other kind of cuspidal sub-representations of π. Finally we achieve the main theorem in this section:

4.8.

Theorem 4.9. *The decomposition of* (π, V) *is presented as follows:*

$$\pi = \bigoplus_{\sigma \in \mathrm{Irr}(\mathrm{GL}_2(F))} \mathrm{Bc}_{K/F}(\sigma),$$

where $\mathrm{Irr}(\mathrm{GL}_2(F))$ *is the set of the classes of the irreducible representations of* $\mathrm{GL}_2(F)$, *and* $Bc_{K/F}$ *is the base change from* $\mathrm{Irr}(\mathrm{GL}_2(F))$ *to* $\mathrm{Irr}(\mathrm{GL}_2(K))$.

Proof. It follows from Proposition 4.8 for non cuspidal representations and the above discussion for cuspidal representations. □

4.9. **Appendix 1.** In the following subsection, we will see how to get the explicit realisation of (π, G) in the vector space $\mathbb{C}[Y_0 \times X_F]$, in other words, the formulas in Proposition 4.2.

(1):

$$[\pi(u(b)F)]\Big(\begin{pmatrix} 0 & 0 \\ \beta & y \end{pmatrix}, \psi\Big)$$

$$= [\rho\Big(i\big(u(b)\big)F\Big)]\Big(\begin{pmatrix} 0 & 0 \\ \beta & y \end{pmatrix}, \psi\Big)$$

$$= [\rho\Big(\begin{pmatrix} m & 0 \\ 0 & m^\vee \end{pmatrix}\begin{pmatrix} 1 & m^{-1}n \\ 0 & 1 \end{pmatrix}\Big)F]\Big(\begin{pmatrix} 0 & 0 \\ \beta & y \end{pmatrix}, \psi\Big)$$

$$= \chi_q^+(det_{X_0} m)[\rho\Big(\begin{pmatrix} 1 & m^{-1}n \\ 0 & 1 \end{pmatrix}\Big)F]\Big(m^{\vee -1}\begin{pmatrix} 0 & 0 \\ \beta & y \end{pmatrix}, \psi\Big)$$

$$= [\rho\Big(\begin{pmatrix} 1 & m^{-1}n \\ 0 & 1 \end{pmatrix}\Big)F]\Big(\begin{pmatrix} 0 & 0 \\ \beta - b^{\sigma^2} y & y \end{pmatrix}, \psi\Big)$$

$$= \psi\Big(\frac{1}{2}\langle m^{-1}n\begin{pmatrix} 0 & 0 \\ \beta - b^{\sigma^2} y & y \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ \beta - b^{\sigma^2} y & y \end{pmatrix}\rangle\Big)F\Big(\begin{pmatrix} 0 & 0 \\ \beta - b^{\sigma} y & y \end{pmatrix}, \psi\Big)$$

$$= \psi\Big(\frac{1}{2}\langle m^{-1}\begin{pmatrix} \mathrm{Tr}_{E/F}(bb^\sigma\beta) - 2\,\mathrm{N}_{K/F}(b)y & b^\sigma\beta^\sigma + b\beta^{\sigma^2} - bb^\sigma y \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ \beta - b^{\sigma^2} y & y \end{pmatrix}\rangle\Big)F\Big(\begin{pmatrix} 0 & 0 \\ \beta - b^{\sigma^2} y & y \end{pmatrix}, \psi\Big)$$

$$= \psi\Big(\frac{1}{2}\langle\begin{pmatrix} \mathrm{N}_{K/F}(b)y - \mathrm{Tr}_{K/F}(bb^\sigma\beta) & b^\sigma\beta^\sigma + b\beta^{\sigma^2} - bb^\sigma y \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ \beta - b^{\sigma^2} y & y \end{pmatrix}\rangle\Big)F\Big(\begin{pmatrix} 0 & 0 \\ \beta - b^{\sigma^2} y & y \end{pmatrix}, \psi\Big)$$

$$= \psi\Big(\mathrm{Tr}_{K/F}(bb^\sigma\beta y) - \mathrm{N}_{K/F}(b)y^2 - \mathrm{Tr}_{K/F}(b\beta\beta^{\sigma^2})\Big)F\Big(\begin{pmatrix} 0 & 0 \\ \beta - b^{\sigma^2} y & y \end{pmatrix}, \psi\Big).$$

(2):

$$[\pi(h(a,d))F]\Big(\begin{pmatrix} 0 & 0 \\ \beta & y \end{pmatrix}, \psi\Big)$$

$$= [\rho\Big(i\big(h(a,d)\big)\Big)F]\Big(\begin{pmatrix} 0 & 0 \\ \beta & y \end{pmatrix}, \psi\Big)$$

$$= [\rho\Big(\begin{pmatrix} m & 0 \\ 0 & n \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \mathrm{N}_{K/F}(ad)^{-1} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \mathrm{N}_{K/F}(ad) \end{pmatrix}\Big)F]\Big(\begin{pmatrix} 0 & 0 \\ \beta & y \end{pmatrix}, \psi\Big)$$

$$= \chi_q^+(det_{X_0}(m))[\rho\Big(\begin{pmatrix} 1 & 0 \\ 0 & \mathrm{N}_{K/F}(ad) \end{pmatrix}\Big)F]\Big(n^{-1}\begin{pmatrix} 0 & 0 \\ \mathrm{N}_{K/F}(ad)\beta & \mathrm{N}_{K/F}(ad)y \end{pmatrix}, \psi\Big)$$

$$= \chi_q^+(\mathrm{N}_{K/F}(ad))[\rho\Big(\begin{pmatrix} 1 & 0 \\ 0 & \mathrm{N}_{K/F}(ad) \end{pmatrix}\Big)F]\Big(\begin{pmatrix} 0 & 0 \\ \frac{\mathrm{N}_{K/F}(ad)}{dd^\sigma a^{\sigma^2}}\beta & \mathrm{N}_{K/F}(a)y \end{pmatrix}, \psi\Big)$$

$$= \chi_q^+(\mathrm{N}_{K/F}(ad))F\left(\begin{pmatrix} 0 & 0 \\ \frac{\mathrm{N}_{K/F}(ad)}{dd^\sigma a^{\sigma^2}}\beta & \mathrm{N}_{K/F}(a)y \end{pmatrix}, \psi^{\mathrm{N}_{K/F}(ad)^{-1}}\right).$$

(3): Assume $K = F(\xi)$. Choose a basis $\mathcal{A} = \{m_0 = e_1 \otimes e_1 \otimes e_1, m_1 = \xi e_1 \otimes e_1 \otimes e_2 + \xi^\sigma e_2 \otimes e_1 \otimes e_1 + \xi^{\sigma^2} e_1 \otimes e_2 \otimes e_1, m_2 = \xi^\sigma e_1 \otimes e_1 \otimes e_2 + \xi^{\sigma^2} e_2 \otimes e_1 \otimes e_1 + \xi e_1 \otimes e_2 \otimes e_1, m_3 = \xi^{\sigma^2} e_1 \otimes e_1 \otimes e_2 + \xi e_2 \otimes e_1 \otimes e_1 + \xi^\sigma e_1 \otimes e_2 \otimes e_1; n_0 = -e_2 \otimes e_2 \otimes e_2, n_1 = \xi e_2 \otimes e_2 \otimes e_1 + \xi^\sigma e_1 \otimes e_2 \otimes e_2 + \xi^{\sigma^2} e_2 \otimes e_1 \otimes e_2, n_2 = \xi^\sigma e_2 \otimes e_2 \otimes e_1 + \xi^{\sigma^2} e_1 \otimes e_2 \otimes e_2 + \xi e_2 \otimes e_1 \otimes e_2, n_3 = \xi^{\sigma^2} e_2 \otimes e_2 \otimes e_1 + \xi e_1 \otimes e_2 \otimes e_2 + \xi^\sigma e_2 \otimes e_1 \otimes e_2\}$ in W_0. Then by Corollary 4.1 (3), we know $i(\omega)(m_i) = n_i$ and $i(\omega)(n_i) = -m_i$ for $0 \leq i \leq 3$. By calculation

$$\begin{pmatrix} \langle m_0, m_0\rangle & \dots & \langle m_0, m_3\rangle & \langle m_0, n_0\rangle & \dots & \langle m_0, n_3\rangle \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \langle m_3, m_0\rangle & \dots & \langle m_3, m_3\rangle & \langle m_3, n_0\rangle & \dots & \langle m_3, n_3\rangle \\ \langle n_0, m_0\rangle & \dots & \langle n_0, m_3\rangle & \langle n_0, n_0\rangle & \dots & \langle n_0, n_3\rangle \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \langle n_3, m_0\rangle & \dots & \langle n_3, m_3\rangle & \langle n_3, n_0\rangle & \dots & \langle n_3, n_3\rangle \end{pmatrix} = \begin{pmatrix} 0 & A \\ -A & 0 \end{pmatrix}$$

where $A = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -\mathrm{Tr}_{K/F}(\xi^2) & -\mathrm{Tr}_{K/F}(\xi\xi^\sigma) & -\mathrm{Tr}_{K/F}(\xi\xi^\sigma) \\ 0 & -\mathrm{Tr}_{K/F}(\xi\xi^\sigma) & -\mathrm{Tr}_{K/F}(\xi^2) & -\mathrm{Tr}_{K/F}(\xi\xi^\sigma) \\ 0 & -\mathrm{Tr}_{K/F}(\xi\xi^\sigma) & -\mathrm{Tr}_{K/F}(\xi\xi^\sigma) & -\mathrm{Tr}_{K/F}(\xi^2) \end{pmatrix}$.

Suppose $A = {}^tP_1P_1$ and $(g_0, \cdots, g_3; h_0, \cdots, h_3) = (m_0, \cdots, m_3; n_0, \cdots, n_3)P$ for some $P = \begin{pmatrix} P_1^{-1} & 0 \\ 0 & P_1^{-1} \end{pmatrix}$. Then:

$$\begin{pmatrix} \langle g_0, g_0\rangle & \dots & \langle g_0, g_3\rangle & \langle g_0, h_0\rangle & \dots & \langle g_0, h_3\rangle \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \langle g_3, g_0\rangle & \dots & \langle g_3, g_3\rangle & \langle g_3, h_0\rangle & \dots & \langle g_3, h_3\rangle \\ \langle h_0, g_0\rangle & \dots & \langle h_0, g_3\rangle & \langle h_0, h_0\rangle & \dots & \langle h_0, h_3\rangle \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \langle h_3, g_0\rangle & \dots & \langle h_3, g_3\rangle & \langle h_3, h_0\rangle & \dots & \langle h_3, h_3\rangle \end{pmatrix} = {}^tP \begin{pmatrix} \langle m_0, m_0\rangle & \dots & \langle m_0, m_3\rangle & \langle m_0, n_0\rangle & \dots & \langle m_0, n_3\rangle \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \langle m_3, m_0\rangle & \dots & \langle m_3, m_3\rangle & \langle m_3, n_0\rangle & \dots & \langle m_3, n_3\rangle \\ \langle n_0, m_0\rangle & \dots & \langle n_0, m_3\rangle & \langle n_0, n_0\rangle & \dots & \langle n_0, n_3\rangle \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \langle n_3, m_0\rangle & \dots & \langle n_3, m_3\rangle & \langle n_3, n_0\rangle & \dots & \langle n_3, n_3\rangle \end{pmatrix} P$$

$$= \begin{pmatrix} {}^tP_1^{-1} & 0 \\ 0 & {}^tP_1^{-1} \end{pmatrix}\begin{pmatrix} 0 & A \\ -A & 0 \end{pmatrix}\begin{pmatrix} P_1^{-1} & 0 \\ 0 & P_1^{-1} \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

i.e. The set $\{g_0, \cdots, g_3; h_0, \cdots, h_3\}$ is a symplectic basis of W_0. Moreover $i(\omega)(g_0, \cdots, g_3; h_0, \cdots, h_3) = (g_0, \cdots, g_3; h_0, \cdots, h_3)P^{-1}\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}P$. And $P^{-1}\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}P = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} = \omega^{-1}_{\mathrm{GSp}(W_0)} \in \mathrm{GSp}(W_0)$ respect to the symplectic basis $\{g_0, \cdots, g_3; h_0, \cdots, h_3\}$.

Now let $\alpha = a_1\xi + a_2\xi^\sigma + a_3\xi^{\sigma^2}, \beta = b_1\xi + b_2\xi^\sigma + b_3\xi^{\sigma^2} \in K$. Put $b = (b_0, \cdots, b_3)$ and $a = (a_0, \cdots, a_3)$. Then

$$[\pi(\omega)F]\left(\begin{pmatrix} 0 & 0 \\ \beta & -b_0 \end{pmatrix}, \psi\right) = \rho[i(\omega)F]\big((n_0, \cdots, n_3)^t b\big)$$

$$= q^{-2} \sum_{(n_0,\cdots,n_3)^t a \in Y_0} F((n_0, \cdots, n_3)^t a, \psi)\psi\big(\langle (n_0, \cdots, n_3)^t a, \omega_{\mathrm{GSp}(W_0)}((n_0, \cdots, n_3)^t b)\rangle\big)$$

$$= q^{-2} \sum_{(n_0,\cdots,n_3)^t a \in Y_0} F((n_0, \cdots, n_3)^t a, \psi)\psi\big(\langle (n_0, \cdots, n_3)^t a, i(\omega^{-1})[(m_0, \cdots, m_3)^t b]\rangle\big)$$

$$= q^{-2} \sum_{(n_0,\cdots,n_3)^t a \in Y_0} F\big(-a_0e_2 \otimes e_2 \otimes e_2 + \alpha e_2 \otimes e_2 \otimes e_1 + \alpha^\sigma e_1 \otimes e_2 \otimes e_2 + \alpha^{\sigma^2} e_2 \otimes e_1 \otimes e_2, \psi\big)$$

$$\psi(\langle -a_0e_2 \otimes e_2 \otimes e_2 + \alpha e_2 \otimes e_2 \otimes e_1 + \alpha^\sigma e_1 \otimes e_2 \otimes e_2 + \alpha^{\sigma^2} e_2 \otimes e_1 \otimes e_2,$$

$$b_0e_1 \otimes e_1 \otimes e_1 + \beta e_1 \otimes e_1 \otimes e_2 + \beta^\sigma e_2 \otimes e_1 \otimes e_1 + \beta^{\sigma^2} e_1 \otimes e_2 \otimes e_1\rangle)$$

$$= q^{-2} \sum_{a_0 \in F, \alpha \in K} F(-a_0e_2 \otimes e_2 \otimes e_2 + \alpha e_2 \otimes e_2 \otimes e_1 + \alpha^\sigma e_1 \otimes e_2 \otimes e_2 + \alpha^{\sigma^2} e_2 \otimes e_1 \otimes e_2, \psi)\psi(a_0b_0 + \mathrm{Tr}_{K/F}(\alpha\beta))$$

$$= q^{-2}\sum_{a_0\in F,\alpha\in K} F\Big(\begin{pmatrix} 0 & 0\\ \alpha & -a_0\end{pmatrix},\psi\Big)\psi(a_0b_0+\mathrm{Tr}_{K/F}(\alpha\beta)).$$

Finally, we obtain

$$[\pi(\omega)F]\Big(\begin{pmatrix} 0 & 0\\ \beta & y\end{pmatrix},\psi\Big) = q^{-2}\sum_{\beta'\in K,y'\in F} F\Big(\begin{pmatrix} 0 & 0\\ \beta' & y'\end{pmatrix},\psi\Big)\psi(yy'+\mathrm{Tr}_{K/F}(\beta\beta')).$$

4.10. **Appendix 2.** We put some calculations for the table in §4.6 in this appendix. From the definition, we see:

$$A\Big(\begin{pmatrix} 0 & 0\\ \beta & y\end{pmatrix},\phi^k\Big) = \sum_{t\in T}\alpha\cdot\pi(t)F_{(0,0;\phi)}\Big(\begin{pmatrix} 0 & 0\\ \beta & y\end{pmatrix},\phi^k\Big)$$

$$= \chi\chi_q^+(\mathrm{N}_{K/F}(ad))\sum_{a,d\in K^\times} F_{(0,0;\phi)}\Big(\begin{pmatrix} 0 & 0\\ \frac{\mathrm{N}_{K/F}(ad)}{dd^\sigma a^{\sigma^2}}\beta & \mathrm{N}_{K/F}(a)y\end{pmatrix},\phi^{\mathrm{N}_{K/F}(ad)^{-1}k}\Big)$$

$$= \sum_{a,d\in K^\times,\mathrm{N}_{K/F}(ad)=k}\chi\chi_q^+(k)F_{(0,0;\phi)}\Big(\begin{pmatrix} 0 & 0\\ \frac{\mathrm{N}_{K/F}(ad)}{dd^\sigma a^{\sigma^2}}\beta & \mathrm{N}_{K/F}(a)y\end{pmatrix},\phi\Big);$$

$$B\Big(\begin{pmatrix} 0 & 0\\ \beta & y\end{pmatrix},\phi^k\Big) = \sum_{t\in T}\alpha\cdot\pi(t)G_{(0,1;\phi)}\Big(\begin{pmatrix} 0 & 0\\ \beta & y\end{pmatrix},\phi^k\Big)$$

$$= \sum_{a,d\in K^\times}\chi\chi_q^+(\mathrm{N}_{K/F}(ad))G_{(0,1;\phi)}\Big(\begin{pmatrix} 0 & 0\\ \frac{\mathrm{N}_{K/F}(ad)}{dd^\sigma a^{\sigma^2}}\beta & \mathrm{N}_{K/F}(a)y\end{pmatrix},\phi^{\mathrm{N}_{K/F}(ad)^{-1}k}\Big)$$

$$= \sum_{a,d\in K^\times,\mathrm{N}_{K/F}(ad)=k}\chi\chi_q^+(k)G_{(0,1;\phi)}\Big(\begin{pmatrix} 0 & 0\\ \frac{\mathrm{N}_{K/F}(ad)}{dd^\sigma a^{\sigma^2}}\beta & \mathrm{N}_{K/F}(a)y\end{pmatrix},\phi\Big);$$

$$(\alpha\cdot\pi(\omega)A)\Big(\begin{pmatrix} 0 & 0\\ \beta & y\end{pmatrix},\phi^k\Big)$$

$$= \sum_{t\in T}\alpha\cdot\pi(t)\big(\alpha\cdot\pi(\omega)F_{(0,0;\phi)}\big)\Big(\begin{pmatrix} 0 & 0\\ \beta & y\end{pmatrix},\phi^k\Big)$$

$$= \sum_{a,d\in K^\times}\chi\chi_q^+(\mathrm{N}_{K/F}(ad))(\alpha\cdot\pi(\omega)F_{(0,0;\phi)})\Big(\begin{pmatrix} 0 & 0\\ \frac{\mathrm{N}_{K/F}(ad)}{dd^\sigma a^{\sigma^2}}\beta & \mathrm{N}_{K/F}(a)y\end{pmatrix},\phi^{k\mathrm{N}_{K/F}(ad)^{-1}}\Big)$$

$$= q^{-2}\sum_{a,d\in K^\times}\chi\chi_q^+(\mathrm{N}_{K/F}(ad))\sum_{\begin{pmatrix} 0 & 0\\ \beta' & y'\end{pmatrix}\in Y_0} F_{(0,0;\phi)}\Big(\begin{pmatrix} 0 & 0\\ \beta' & y'\end{pmatrix},\phi^{k\mathrm{N}_{K/F}(ad)^{-1}}\Big)\phi^{k\mathrm{N}_{K/F}(ad)^{-1}}\Big(\mathrm{N}_{K/F}(a)yy'+\mathrm{Tr}_{K/F}(\frac{\mathrm{N}_{K/F}(ad)}{dd^\sigma a^{\sigma^2}}\beta\beta')\Big)$$

$$= q^{-2}\sum_{a,d\in K^\times,\mathrm{N}_{K/F}(ad)=k}\chi\chi_q^+(k) = q^{-2}\chi\chi_q^+(k)(q^3-1)(q^2+q+1);$$

Notice: $G_{(0,1;\phi)}(\begin{pmatrix} 0 & 0\\ \beta & 1\end{pmatrix},\phi) = \phi(-\mathrm{N}_{K/F}(\beta))$ by the formula (22).

$$(\alpha\cdot\pi(\omega)B)\Big(\begin{pmatrix} 0 & 0\\ \beta & y\end{pmatrix},\phi^k\Big) = \sum_{t\in T}\alpha\cdot\pi(t)\alpha\cdot\pi(\omega)G_{(0,1;\phi)}(\begin{pmatrix} 0 & 0\\ \beta & y\end{pmatrix},\phi^k)$$

$$= \sum_{a,d\in K^\times}\chi\chi_q^+(\mathrm{N}_{K/F}(ad))\alpha\cdot\pi(\omega)G_{(0,1;\phi)}\Big(\begin{pmatrix} 0 & 0\\ \frac{\mathrm{N}_{K/F}(ad)}{dd^\sigma a^{\sigma^2}}\beta & \mathrm{N}_{K/F}(a)y\end{pmatrix},\phi^{k\mathrm{N}_{K/F}(ad)^{-1}}\Big)$$

$$= q^{-2}\sum_{a,d\in K^\times}\chi\chi_q^+(\mathrm{N}_{K/F}(ad))\sum_{\begin{pmatrix} 0 & 0\\ \beta' & y'\end{pmatrix}\in Y_0} G_{(0,1;\phi)}\Big(\begin{pmatrix} 0 & 0\\ \beta' & y'\end{pmatrix},\phi^{k\mathrm{N}_{K/F}(ad)^{-1}}\Big)\phi^{k\mathrm{N}_{K/F}(ad)^{-1}}\Big(\mathrm{N}_{K/F}(a)yy'+\mathrm{Tr}_{K/F}(\frac{\mathrm{N}_{K/F}(ad)}{dd^\sigma a^{\sigma^2}}\beta\beta')\Big)$$

$$= q^{-2} \sum_{a,d\in K^\times, \mathrm{N}_{K/F}(ad)=k} \chi\chi_q^+(k) \sum_{\beta'\in K} \phi(-\mathrm{N}_{K/F}(\beta'))\phi\Big(\mathrm{N}_{K/F}(a)y + \mathrm{Tr}_{K/F}(aa^\sigma d^{\sigma^2}\beta\beta')\Big);$$

(1)

$$A(x_{00}) = \sum_{t\in T} \Big(\alpha\cdot\pi(t)F_{(0,0;\phi)}\Big)(x_{00})$$

$$= \sum_{t=\begin{pmatrix} a & 0\\ 0 & d\end{pmatrix}\in T} \chi\chi_q^+(\mathrm{N}_{K/F}(ad))F_{(0,0,\phi)}(\begin{pmatrix} 0 & 0\\ 0 & 0\end{pmatrix}, \phi^{\mathrm{N}_{K/F}(ad)^{-1}})$$

$$= \sum_{t=\begin{pmatrix} a & 0\\ 0 & d\end{pmatrix}\in T; \mathrm{N}_{K/F}(ad)=1} 1 = (q^3-1)(q^2+q+1)$$

$$A(x_{10}) = A(x_{01}) = A(y_k) = 0.$$

(2)

$$B(x_{00}) = \sum_{t\in T} \alpha\cdot\pi(t)G_{(0,1;\phi)}(\begin{pmatrix} 0 & 0\\ 0 & 0\end{pmatrix}, \phi) = 0 = B(x_{1,0})$$

$$B(x_{01}) = \sum_{t\in T} \alpha\cdot\pi(t)G_{(0,1;\phi)}(\begin{pmatrix} 0 & 0\\ 0 & 1\end{pmatrix}, \phi)$$

$$= \sum_{a,d\in K^\times} G_{(0,1;\phi)}\Big(\begin{pmatrix} 0 & 0\\ 0 & \mathrm{N}_{K/F}(a)\end{pmatrix}, \phi^{\mathrm{N}_{K/F}(ad)^{-1}}\Big)$$

$$= \sum_{a,d\in K^\times, \mathrm{N}_{K/F}(a)=\mathrm{N}_{K/F}(d)=1} \chi\chi_q^+(\mathrm{N}_{K/F}(ad)) = (q^2+q+1)^2$$

$$B(y_k) = \sum_{t\in T} \alpha\cdot\pi(t)G_{(0,1,\phi)}(y_k)$$

$$= \sum_{a,d\in K^\times} \chi\chi_q^+(\mathrm{N}_{K/F}(ad))G_{(0,1;\phi)}\Big(\begin{pmatrix} 0 & 0\\ \frac{\mathrm{N}_{K/F}(ad)}{dd^\sigma a^{\sigma^2}} & \mathrm{N}_{K/F}(a)\end{pmatrix}, \phi^{k\,\mathrm{N}_{K/F}(ad)^{-1}}\Big)$$

$$= \sum_{a,d\in K^\times, \mathrm{N}_{K/F}(a)=1, \mathrm{N}_{K/F}(d)=k} \chi\chi_q^+(k)G_{(0,1;\phi)}\Big(\begin{pmatrix} 0 & 0\\ aa^\sigma d^{\sigma^2} & 1\end{pmatrix}, \phi\Big)$$

$$\overset{\text{the equality (22)}}{=} \sum_{a,d\in K^\times, \mathrm{N}_{K/F}(a)=1, \mathrm{N}_{K/F}(d)=k} \chi\chi_q^+(k)\phi\Big(-\mathrm{N}_{K/F}(aa^\sigma d^{\sigma^2})\Big)$$

$$= \sum_{a,d\in K^\times, \mathrm{N}_{K/F}(a)=1, \mathrm{N}_{K/F}(d)=k} \chi\chi_q^+(k)\phi(-k)$$

$$= \phi(-k)\chi\chi_q^+(k)(q^2+q+1)^2.$$

(3)

$$(\alpha\cdot\pi(\omega)A)(x_{00}) = \alpha\cdot\pi(\omega)A(X_{10}) = \alpha\cdot\pi(\omega)A(x_{01}) = q^{-2}(q^3-1)(q^2+q+1)$$

$$\alpha\pi(\omega)A(y_k) = \chi\chi_q^+(k)q^{-2}(q^3-1)(q^2+q+1).$$

(4)

$$\alpha\cdot\pi(\omega)B(x_{00}) = q^{-2} \sum_{a,d\in K^\times, \mathrm{N}_{K/F}(ad)=1, \beta'\in K} \phi(-\mathrm{N}_{K/F}(\beta')) = q^{-2}(q^3-1)(q^2+q+1)(-q^2-q)$$

(Since $\sum_{\beta'\neq 0}\phi(-\mathrm{N}_{K/F}(\beta')) + q^2+q+1 = 0$, we have $\sum_{\beta\in K}\phi(-\mathrm{N}_{K/F}(\beta)) = -q^2-q$).

$$\alpha\cdot\pi(\omega)B(x_{10}) = q^{-2} \sum_{a,d\in K^\times, \mathrm{N}_{K/F}(ad)=1} \sum_{\beta'\in K} \phi(-\mathrm{N}_{K/F}(\beta'))\phi(\mathrm{Tr}_{K/F}(\frac{1}{dd^\sigma a^{\sigma^2}}\beta')$$

$$= q^{-2} \sum_{a,d\in K^\times, \mathrm{N}_{K/F}(ad)=1, \beta'\in K} \phi(-\mathrm{N}_{K/F}(dd^\sigma a^{\sigma^2}\beta'))\phi(\mathrm{Tr}_{K/F}(\beta'))$$

$$= q^{-2} \sum_{a,d\in K^\times, \mathrm{N}_{K/F}(ad)=1, \beta'\in K} \phi(-\mathrm{N}_{K/F}(dd^{\sigma}a^{\sigma^2}\beta'))\phi(\mathrm{Tr}_{K/F}(\beta'))$$
$$= q^{-2} \sum_{\beta'\in K} \sum_{a,d\in K^\times, \mathrm{N}_{K/F}(ad)=1} \phi(-\mathrm{N}_{K/F}(d)\,\mathrm{N}_{K/F}(\beta'))\phi(\mathrm{Tr}_{K/F}(\beta')) \cdots (\star)$$

(i) If $\beta' = 0$, $\sum_{a,d\in K^\times, \mathrm{N}_{K/F}(ad)=1} \phi(-\mathrm{N}_{K/F}(d)\,\mathrm{N}_{K/F}(\beta') + \mathrm{Tr}_{K/F}(\beta')) = \sum_{a,d\in K^\times, \mathrm{N}_{K/F}(ad)=1} 1 = (q^3-1)(q^2+q+1)$;
(ii) If $\beta' \neq 0$,

$$\sum_{a,d\in K^\times, \mathrm{N}_{K/F}(ad)=1} \phi(-\mathrm{N}_{K/F}(d)\,\mathrm{N}_{K/F}(\beta'))$$
$$= \sum_{l\in K^\times, \mathrm{N}_{K/F}(l)=1} \sum_{d\in K^\times} \phi(-\mathrm{N}_{K/F}(d)\,\mathrm{N}_{K/F}(\beta'))$$
$$= (q^2+q+1)(-q^2-q-1) = -(q^2+q+1)^2,$$

so

$$(\star) = q^{-2}[(q^3-1)(q^2+q+1) + \sum_{\beta'\in K^\times} -(q^2+q+1)^2\phi(\mathrm{Tr}_{K/F}(\beta'))]$$
$$= q^{-2}[(q^2+q+1)(q^3-1) + (q^2+q+1)^2] = q^{-2}(q^2+q+1)^2 q = q^{-1}(q^2+q+1)^2.$$

$$\alpha\cdot\pi(\omega)B(x_{01}) = q^{-2} \sum_{a,d\in K^\times, \mathrm{N}_{K/F}(ad)=1} \sum_{\beta'\in K} \phi(-\mathrm{N}_{K/F}(\beta'))\phi(\mathrm{N}_{K/F}(a))$$
$$= q^{-2} \sum_{a,d\in K^\times, \mathrm{N}_{K/F}(ad)=1} \phi(\mathrm{N}_{K/F}(a))(-q^2-q)$$
$$= q^{-2}(q^2+q+1)(-q^2-q-1)(-q^2-q) = q^{-1}(q+1)(q^2+q+1)^2.$$

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